Certain Algebraic Procedures for the Aperiodic Stability analysis and Counting the Number of Complex Roots of Linear Systems

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Abstract

To evaluate the performance of a linear time-invariant system, various measures are available. In this paper employing Routh's table, two geometrical criteria for the aperiodic stability analysis of linear time-invariant systems having real coefficients are formulated. The proposed algebraic stability criteria check whether the given linear system is aperiodically stable or not.The additional significance of the two criteria is , it can be used to count the number of complex roots of a system having real coefficients which is not possible by the use of original Routh's Table. These procedures can also be used for the design of linear systems. In the proposed methods,the characteristic equation having real coefficients are first converted to complex coefficient equations using Romonov's transformation. These complex coefficients are used in two different ways to form the Modified Routh's tables for the two schemes named as Sign Pair Criterion I (SPC I) and Sign Pair Criterion II (SPC II). It is found that the proposed algorithms offer computational simplicity compared to other algebraic methods and is illustrated with suitable examples. The developed MATLAB program make the analysis most simple.

Keywords: Complex Roots of a Polynomial, Linear Systems , Aperiodic Stability Analaysis, Modified Routh's table, Sign pair criterion I and II.

1. INTRODUCTION

In the case of a linear invariant continuous system, the stability analysis can be carried out by the knowledge of root distribution of it's characteristic equation. To analyse the stability of complex polynomials the generalized Routh-Hurwitz method was investigated in [1] - [6]. Frank [1] and Agashe [2] developed a new Routh like algorithm to determine the number of RHP roots in the complex case. Benidir and Picinbono [3] proposed an extended Routh table which considers singular cases of vanishing leading array element. By adding intermediate rows in the Routh array, Shyan and Jason [4] developed a tabular column ,which is also a complicated one. Adel [5] has done the stability analysis of complex polynomials using the J-fraction expansion , Hurwitz Matrix determinant and also generalized Routh's Array. Formation of Routh's Table by retaining the 'j' terms of the complex coefficients and the stability analysis using Sign Pair Criterion I were done in [6].

Information about the aperiodic stability of a control system is of paramount importance for any design problem. This is generally used for the design of instrumentation systems, network analysis and automatic controls. The existence of real and distinct roots in the negative real axis determine the aperiodic behavior of a linear system. The presence of any complex roots shows that the system is aperiodically unstable. To analyze the aperiodic stability , a generalized method was investigated in [7] by Fuller. Romonov [8] developed a new transformation to determine the aperiodic behavior of the linear system which results in complex coefficient polynomials. A popular but cumbersome method for determining the number of real roots in polynomial with real coefficients is by Sturm's theorem [9]. Itzhack presented [10] a three-steps transformation procedure which develops a polynomial whose number of right hand plane poles equals the number of complex roots present in the original polynomial .

In the proposed methods ,the characteristic equation having real coefficients are first converted to complex coefficient equations using Romonov's transformation. These complex coefficients are used in two different ways to form the Modified Routh's tables for the two schemes named as Sign Pair Criterion I (SPC I) and Sign Pair Criterion II (SPC II). The beauty of the Routh's algorithm lies in finding the aperiodic stability of the system without determining the roots of the system. In the first approach, formation of Routh's Table is done by retaining the 'j' terms of the complex coefficients and the stability analysis is done using Sign Pair Criterion I (SPC I) . The proof is given in [6]. In the second scheme, a geometrical procedure is presented which is named as Sign Pair Criterion II (SPC II) and is formulated with the help of 'Modified Routh's table' after separating the real and imaginary parts of the characteristic equation by substituting s='jω'. Applying Routh–Hurwitz criterion, the number of the roots of F(s)=0 having positive real part can be revealed. Then by the use of the proposed algorithm, the aperiodic behavior of linear systems, and also the number of complex roots of the characteristic equations can be determined. MATLAB Program is developed for the proposed schemes and aperiodic stability is analyzed in a most simple way regardless of the order of the system .The computational simplicity is illustrated with examples.

2. APERIODIC STABILITY ANALYSIS

A linear time invariant control system represented by the characteristic equation F(s) =0, with real coefficients is aperiodically stable only when it's all roots are distinct ,real and lie on the negative real axis of 's' plane. To analyse this situation, Romonov [8] suggested a transformed polynomial of F(s) into a complex polynomial defined as given in equation (1) .

$$F'(s) = [F(s)_{s=js} + j(\frac{dF(js)}{d(js)})]$$
$$= F(js) + j(\frac{dF(js)}{d(js)}) \qquad (1)$$

Applying Routh–Hurwitz criterion, the number of the roots of F(s)=0 having positive real part can be revealed. After the transformation, the real coefficient polynomial is converted to complex coefficient polynomial and the two proposed schemes SPC I and SPC II can be used for the aperiodic stability analysis and each sign pair which fails to satisfy the condition for stability represents the existence of two complex roots (one complex conjugate pair) and this leads to aperiodic instability. These procedures can also be used for the design of linear systems.

3. PROPOSED PROCEDURES

2.1. Sign Pair Criterion I (SPC I)

Let F(s)=0 be the nth degree characteristic equation of a linear time invariant system and written as

$$F(s) = s^n + (a1 + jb1)s^{n-1} + (a2 + jb2)s^{n-2} + ... + (an + jbn) = 0 \qquad (2)$$

Where $(a_i + jb_i)$ are the complex coefficients. The first two rows of Modified Routh's Table for the equation (2) are shown as below.

$$\begin{array}{cccc} 1 & jb_1 & a_2 & jb_3..... \\ a_1 & jb_2 & a_3 & jb_4..... \end{array}$$

Applying Routh multiplication rule ,the complete table with '2n' number of rows are formed.

$$\begin{array}{cccc} 1 & jb_1 & a_2 & jb_3..... \\ a_1 & jb_2 & a_3 & jb_4..... \\ jc_1 & c_2 & jc_3.... & \\ jd_1 & d_2 & jd_3.... & \\ . & . & ..... & \\ . & . & ..... & \\ g_1 & & & \\ h_1 & & & \end{array}$$

Using the above Table, pairs are formed using the first column and starting from the first row.

$P_1 = (1, a_1), \quad P_2 = (jc_1, jd_1), \quad Pn = (g_1, h_1)$

It is ascertained that the two elements in each pair has to maintain same sign for the system to be stable. Proof is given in [6].

2.2. Sign Pair Criterion II (SPC II)

In this approach , the characteristic equation given in 's' domain is converted to frequency domain by replacing s='jω' and the real and imaginary parts are separated . The coefficients of real parts are used to form the first row of 'Modified Routh's table' and the coefficients of imaginary parts are the elements of second row of the table., By applying the normal Routh multiplication rule, the complete Routh's Array is formed with '2n+1' number of rows ,where 'n' is the order of the system**.**

2.2.1.Algorithm for the proposed approach

1. Get F(s)=0 with complex coefficients.
2. With s=jω, form F(jω) = R(ω)+jI(ω)
3. Use the coefficients of R(ω) & I(ω), form the first and second rows of Routh's table.
4. If the first element in the first row is negative , multiply the full row elements by -1.
5. If the first element in the second row is zero, interchange first and second row.
6. Follow the Common Routh's multiplication rule to get the complete table with '2n+1' rows.
7. Get 'n' sign pairs using the first column elements starting from second row.

Consider the nth degree characteristic equation F(S)=0 of a linear time invariant system with complex coefficients,

$$F(s) = s^n + (A1 + jB1)s^{n-1} + (A2 + jB2)s^{n-2} + ...+ (An + jBn) = 0 \qquad (3)$$

Where $Ai + jBi$ are the complex coefficients. By substituting s='jω' and separating real and imaginary parts, the characteristic equation can be written as follows.

$$\begin{aligned} F(j\omega) &= R(\omega) + jI(\omega) = 0 \\ &= a1\omega^n + a2\omega^{n-1} + a3\omega^{n-2} + ... + j(b1\omega^n + b2\omega^{n-1} + b3\omega^{n-2}......) \end{aligned} \qquad (4)$$

Using the coefficients of real and imaginary parts, the first two rows of Modified Routh's Table is formed as

$$\begin{array}{cccc} a_1 & a_2 & a_3 & a_4..... \\ b_1 & b_2 & b_3 & b_4..... \end{array}$$

The direct Routh's multiplication rule is applied and the complete Modified Routh's Table with '2n+1'number of rows is formed as

$$\begin{array}{cccc} a1 & a2 & a3 & a4..... \\ b1 & b2 & b3 & b4..... \\ c1 & c2 & c3.... & \\ d1 & d2 & d3.... & \\ e1 & e2 & ..... & \\ . & . & ..... & \\ g1 & & & \\ h1 & & & \end{array}$$

From the elements of first column of the above table, starting from the second row, the following pairs may be grouped respectively: $P_1 = (b_1, c_1)$, $P_2 = (d_1, e_1)$, $Pn = (g_1, h_1)$. It is ascertained that the two elements in each pair has to maintain same sign for all the roots of F(s)=0 to be real and distinct. If one pair fails to satisfy this condition, it is inferred that there exists two numbers of complex roots (one complex conjugate pair) for F(s)=0 and the system is aperiodically unstable.

4. ILLUSTRATIONS

4.1 Example 1

$$F(s) = s^2 + 3s + 2 = 0 \tag{5}$$

As per equation (1)

$$F(js) = (js)^2 + 3(js) + 2 = 0$$

$$F'(s) = (js)^2 + 3(js) + 2 + j[2(js) + 3]$$

$$F'(s) = s^2 + (2 - j3)s + (-2 - j3) \tag{6}$$

4.1.1 Modified Routh's Table using SPC I

$$\begin{array}{ccc} 1 & -j3 & -2 \\ 2 & -j3 & \\ -j1.5 & -2 & \\ -j0.33 & & \end{array}$$

The Sign Pairs are P_1 = (+1, + 2) and P_2 = (-j1.5, -j0.33).It is noted that the two elements in each pair have the same sign and obey SPC-I. Hence the system is aperiodically stable.

4.1.2 Modified Routh's Table using SPC II

The equation (6) can be written as

$$F'(j\omega) = (-1\omega^2 + 3\omega - 2) + j(2\omega - 3) = 0$$

$$= R(\omega) + jI(\omega) = 0 \tag{7}$$

Using F(jω), the Modified Routh's Table is formed as

$$\begin{array}{ccc} 0 & +2 & -3 \\ +1 & -3 & +2 \\ +2 & -3 & \\ -1.5 & +2 & \\ -0.33 & & \end{array}$$

The Sign Pairs are formed as P_1 = (+1, + 2) and P_2 = (-1.5, -0.33). It is noted that the two elements in each pair have the same sign and obey SPC-I. Hence the system is aperiodically stable. The roots of F(s) are found as -1 and -2 which are real values ; this verifies the result.

4.2 Example 2

$$F(s) = s^3 + 6s^2 + 11s + 6 = 0 \tag{8}$$

$$F'(s) = (js)^3 + 6(js)^2 + 11(js) + 6 + j[3(js)^2 + 12(js) + 11]$$

$$F'(s) = s^3 + (3 - j6)s^2 + (-11 - j12)s + (-11 + j6) \tag{9}$$

4.2.1 Modified Routh's Table using SPC I

$$\begin{array}{cccc} +1 & -j6 & -11 & +j6 \\ +3 & -j12 & -11 & \\ -j2 & -7.33 & +j6 & \\ -j1 & -2 & & \\ -3.33 & +j6 & & \\ -0.2 & & & \end{array}$$

Sign Pairs are P_1 = (+1, + 3) , P_2 = (-j2, -j1) and P_3 = (-3.33, -0. 2) . All pairs obey SPC-I.

4.2.2 Modified Routh's Table using SPC II

Equation (9) can be written as

$$F'(j\omega) = (-3\omega^2 + 12\omega - 11) + j(-\omega^3 + 6\omega^2 - 11\omega + 6) = 0 \tag{10}$$
$$= R(\omega) + jI(\omega) = 0$$

Modified Routh's Table is formed as

$$\begin{array}{cccc} 0 & -3 & +12 & -11 \\ -1 & +6 & -11 & +6 \\ -3 & +12 & -11 & \\ +2 & -7.3 & +6 & \\ +1 & -2 & & \\ -3.3 & +6 & & \\ -0.2 & & & \end{array}$$

Sign Pairs are are P_1 = (-1, - 3) , P_2 = (+2, +1) and P_3 = (-3.33, -0. 2) . All pairs obey SPC-I. The system is aperiodically stable. The roots of F(s) = 0 are -1, -2 and -3 which are real values and the result is verified.

4.3 Example 3

$$F(s) = s^2 + 2s + 2 = 0 \tag{11}$$

The equation (11) can be written as

$$F'(s) = (js)^2 + 2(js) + 2 + j[2(js) \;\; + 2]$$
$$F'(s) = s^2 + (2 - j2)s + (-2 - j2) \tag{12}$$

4.3.1 Modified Routh's Table using SPC I

$$\begin{array}{ccc} 1 & -j2 & -2 \\ 2 & -j2 & \\ -j1 & -2 & \\ +j2 & & \end{array}$$

Pairs are formed as P_1 = (+1, +3) , P_2 = (-j1, +j2) and P_2 does not obey SPC I .

4.3.2 Modified Routh's Table using SPC II

The equation (12) can be written as

$$F'(j\omega) = (-1\omega^2 + 2\omega - 2) + j(2\omega - 2) = 0$$
$$= R(\omega) + jI(\omega) = 0 \tag{13}$$

The Modified Routh's Table is formed as

$$\begin{array}{ccc} 0 & +2 & -2 \\ +1 & -2 & +2 \\ +2 & -2 & \\ -1 & +2 & \\ +2 & & \end{array}$$

P_1 = (+1, +2) , P_2 = (-1, +2) and P_2 fails to obey SPC II. Hence the system is aperiodically unstable and there exists 2 numbers of complex roots for the equation F(s)=0. The roots of the equation are -1+j1 and -1-j1, which verifies the result.

4.4 Example 4 [4]

$$F(s) = s^3 + 5s^2 + 8s + 6 = 0 \tag{14}$$

$$F'(s) = (js)^3 + 5(js)^2 + 8(js) + 6 + j[3(js)^2 + 10(js) + 8]$$
$$F'(s) = s^3 + (3 - j5)s^2 + (-8 - j10)s + (-8 + j6) \tag{15}$$

4.4.1 Modified Routh's Table using SPC I

$$\begin{array}{cccc} +1 & -j5 & -8 & +j6 \\ +3 & -j10 & -8 & \\ -j1.67 & -5.33 & +j6 & \\ -j0.4 & +2.8 & & \\ -17 & +j6 & & \\ +2.94 & & & \end{array}$$

P_1 = (+1, + 3) , P_2 = (-j1.67, -j0.4) and P_3 = (-17, +2.94) and P_3 fails to obey SPC.

4.4.2 Modified Routh's Table using SPC II

$$\begin{array}{llll} 0 & -3 & +10 & -8 \\ -1 & +5 & -8 & +6 \\ -3 & +10 & -8 & \\ +1.67 & -5.3 & +6 & \\ +0.4 & +2.8 & & \\ -17 & +6 & & \\ +2.94 & & & \end{array}$$

P_1 = (-1, - 3) , P_2 = (+1.67, +0.4) and P_3 = (-17, +2.94) and P_3 fails to obey SPC II. It shows the existence of two complex roots as given in [4] and the system is aperiodically unstable. The roots of the equation (15) are -3, -1 + j1 and -1 – j1 which verifies the result.

4.5 Example 5 [4]

$$F(s) = s^2 + Ks + 2 = 0 \tag{17}$$

Design the value of 'K' for the system to be aperiodic stable.

$$F'(s) = (js)^2 + K(js) + 2 + j[2(js) + K]$$
$$F'(s) = s^2 + (2 - jK)s + (-2 - jK) \tag{18}$$

4.5.1 Modified Routh's Table using SPC I

$$\begin{array}{lll} 1 & -jK & -2 \\ 2 & -jK & \\ -j(0.5K) & -2 & \\ +j(8-K^2) & & \end{array}$$

For the system to be aperiodic stable , (0.5K) > 0 and 8 – K^2 < 0 to get the two elements of P_2 with same sign (-ve). ie; K must be greater than square root of 8. K > 2.82. The design is verified with the result of [4].

4.5.2 Modified Routh's Table using SPC II

The equation (18) can be written as

$$F'(j\omega) = (-\omega^2 + K\omega - 2) + j(2\omega - K) = 0$$
$$= R(\omega) + jI(\omega) = 0 \tag{19}$$

The Modified Routh's Table is formed as

$$\begin{array}{lll} 0 & +2 & -K \\ +1 & -K & +2 \\ +2 & -K & \\ -0.5K & +2 & \\ -(0.5K^2-4)/0.5K & & \end{array}$$

For the system to be aperiodic stable , 0.5K > 0 and (0.5K^2- 4) > 0, which gives the same result as given in [4]. The condition for aperiodic stability is K > 2.82.

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