Sine and Cosine Fresnel Transforms

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Abstract

Two novel transforms, related together and called Sine and Cosine Fresnel Transforms, as well as their optical implementation are presented. Each transform combines both backward and forward light propagation in the framework of the scalar diffraction approximation. It has been proven that the Fresnel transform is the optical version of the fractional Fourier transform. Therefore the former has the same properties as the latter. While showing properties similar to those of the Fresnel transform and therefore of the fractional Fourier transform, each of the Sine and Cosine Fresnel transforms provides a real result for a real input distribution. This enables saving half of the quantity of information in the complex plane. Because of parallelism, optics offers high speed processing of digital signals. Speech signals should be first represented by images through special light modulators for example. The Sine and Cosine Fresnel transforms may be regarded respectively as the fractional Sine and Cosine transforms which are more general than the Cosine transform used in information processing and compression.

Keywords: Diffraction, Fresnel Transform, Fractional Fourier Transform, Cosine Transform, Sine Transform.

1. INTRODUCTION

The fractional order Fourier transform shows interesting properties [1] and has been used in various domains [2-5] including information processing and compression [3,4]. This transform is very useful in both computer-based (including DSP and FPGA) or optics-based signal processing. The fractional Fourier transform is more general than the Fourier transform since the former transforms a time (or space) domain signal into a time-frequency domain in the same way. In other words, fractional Fourier domains correspond to oblique axes in the time-frequency plane.

An optical implementation of the fractional Fourier transform has been proposed [5-7]. It has been proven that the Fresnel transform as defined by reference [1] is the formulation of the optical version of the fractional Fourier transform [6,7].

The Fourier transform of a real distribution is generally complex. A particular case occurs when the initial real distribution is even, since such a distribution is Fourier transformed into an even real distribution. However the Cosine transform, which is derived from Fourier transform, transforms any real distribution into a real distribution (spectrum). For this reason, the Cosine transform is used more than the Fourier transform in signal and image compression. The famous JPEG format is a typical example.

The Fresnel transform generally transforms real distributions, no matter they are even or not, into complex distributions. In this paper, we will present a new transform derived from the Fresnel transform that behaves like the Cosine transform with respect to real distributions. This transform is referred to as the Cosine Fresnel transform. In a similar way, we will define the Sine Fresnel transform.

While showing properties similar to those of the Fresnel transform and therefore to those of the fractional Fourier transform, each of the Sine and Cosine Fresnel transforms provides a real result for real input distributions. This enables saving half of the quantity of information. The Sine and Cosine Fresnel transform may be regarded respectively as the fractional Sine and Cosine transforms which are more general than the Cosine transform used in information processing and compression. In other words, the main interest of the two new transforms, proposed in the present paper, lies in the fact that they combine the advantages of the fractional Fourier transform with the advantages of the Cosine transform.

We also present an optical implementation of these two transforms. The advantage of optics is the high speed of processing of digital signals. Although derived from the same transform, namely the Fresnel transform, and the only mathematical difference lies in using sine and cosine functions respectively, the Sine and Cosine Fresnel transforms show some distinct behaviors that we will expose in the discussion.

The remainder of the paper will be organized as follows: After the present introduction, section 2 will be devoted to the Fresnel transform. Then, the Cosine Fresnel transform will be defined in section 3, followed by the Sine Fresnel transform. In section 4, the optical implementation of these transforms will be addressed. The section 5 presents some experimental illustrations that will be used in the discussion of section 6. Concluding remarks will be given in section 7.

2. FRESNEL TRANSFORM

The Fresnel transform models diffraction in the Fresnel zone as well as in the Fraunhofer zone. Hence, the diffraction field observed at a distance z behind an object is [1]:

$$h(x,z) = \frac{\exp\left(i2\pi z/\lambda\right)\exp(-i\pi/4)}{\sqrt{\lambda z}} h(x) * f_k(x,z) \tag{1}$$

For brevity of notation, the analysis is limited to the one-dimensional consideration. $h(x)=h(x,z=0)$ is the initial field just behind the object, and * denotes convolution. The Fresnel kernel is expressed as follows:

$$f_k(x,z) = \exp\left(j\pi \frac{x^2}{\lambda z} \right) \tag{2}$$

Also for brevity of notation, the constant term of propagation $\exp\left(j2\pi z/\lambda\right)$ and the constant phase shift factor $\exp(-j\pi/4)/\sqrt{\lambda z}$ will be ignored, λ is the illuminating wavelength. Equation (1) results from an assumption that enables treating light as a scalar phenomenon where the components of either the electric or magnetic field can be treated independently in a similar way [8].

The convolution form of Equation (1) is expanded as follows:

$$h(x,z) = \exp\left(j\frac{\pi}{\lambda z} x^2 \right) \int_{-\infty}^{+\infty} \exp\left(j\frac{\pi}{\lambda z} x_1^2 \right) h(x_1, 0) \exp\left(-2j\frac{\pi}{\lambda z} x_1 x \right) dx_1 \tag{3}$$

3. SINE AND COSINE FRESNEL TRANSFORMS

Although $h(x)$ may be real, $h(x,z)$ is generally not real because the Fresnel kernel is complex. The idea is to propose a new kernel by taking only the real part of the Fresnel kernel:

$$f_C(x,z) = \frac{f_k(x,z) + f_k(x,-z)}{2} = \cos\left(\pi \frac{x^2}{\lambda z} \right) \tag{4}$$

This leads to a new transform that we call Cosine Fresnel transform. The result $h_C(x,z)$, observed at a distance z, is expressed as follows:

$$h_C(x,z) = h(x) * f_C(x,z) = \frac{h(x,z) + h(x,-z)}{2} \tag{5}$$

Thus, the Cosine Fresnel transform is the average of the diffraction field at distance z (wave moving forwards) and the diffraction field at distance $-z$ (wave moving backwards). Therefore, the Cosine Fresnel transform has similar properties as the Fresnel transform itself. The advantage is that Cosine Fresnel transform is real for an initial real distribution. Analogy may be done with the relationship between the Fourier transform and the Cosine transform. Since the Fourier transform is complex, the Cosine transform, which is real, is used more than the former in image compression. The famous JPEG format is a typical example.

Similarly we can define the Sine Fresnel transform $h_S(x,z)$, observed at a distance z , as follows:

$$h_S(x,z) = h(x) * f_S(x,z) = \frac{h(x,z) - h(x,-z)}{2j} \tag{6}$$

With

$$f_S(x,z) = \frac{f_k(x,z) - f_k(x,-z)}{2j} = \sin\left(\pi \frac{x^2}{\lambda z}\right) \tag{7}$$

Let us express the Cosine Fresnel transform in the spatial frequency domain by Fourier transforming Equation (5):

$$H_C(u,z) = H(u,0) \times F_C(u,z) \tag{8}$$

where $H(u,0)$ is the Fourier transform of the initial field $h(x,0)$, $H_C(u,z)$ is the Fourier transform of the Cosine Fresnel transform $h_C(x,z)$, and $F_C(u,z)$ is the Fourier transform of the Cosine Fresnel kernel of Equation (4). By looking at the table of Fourier transform pairs, for example in [1], we find:

$$F_C(u,z) = \sqrt{\lambda z}\cos\left(-\pi\lambda z u^2 + \pi/4\right) \tag{9}$$

For the Sine Fresnel transform, we also obtain an even real function, namely:

$$F_S(u,z) = \sqrt{\lambda z}\sin\left(-\pi\lambda z u^2 + \pi/4\right) \tag{10}$$

The phase shift factor of $\pi/4$ as well as the multiplicative factor $\sqrt{\lambda z}$ in Equations (9) and (10) are in fact compensated by respectively the real and imaginary part of the constant phase shift factor that we ignored above, namely $\exp(-j\pi/4)/\sqrt{\lambda z}$. The remaining terms of Equations (9) and (10), namely $\cos\left(-\pi\lambda z u^2\right)$ and $\sin\left(-\pi\lambda z u^2\right)$ respectively, show interesting properties. For example, self-imaging related properties will be detailed in a future publication with some applications.

We can also express the Sine and Cosine Fresnel transform in a form similar to Equation (3):

$$h_C(x,z) = \int_{-\infty}^{+\infty} h(x_1,0)\cos\left(\frac{\pi}{\lambda z}(x-x_1)^2\right)dx_1 \tag{11}$$

and

$$h_S(x,z) = \int_{-\infty}^{+\infty} h(x_1,0)\sin\left(\frac{\pi}{\lambda z}(x-x_1)^2\right)dx_1 \tag{12}$$

The Sine and Cosine Fresnel transforms are equivalent to the fractional Fourier transform since both of them transform a time (or space) domain signal into a time-frequency domain in the same way. In other words, fractional Fourier domains correspond to oblique axes in the time-frequency plane.

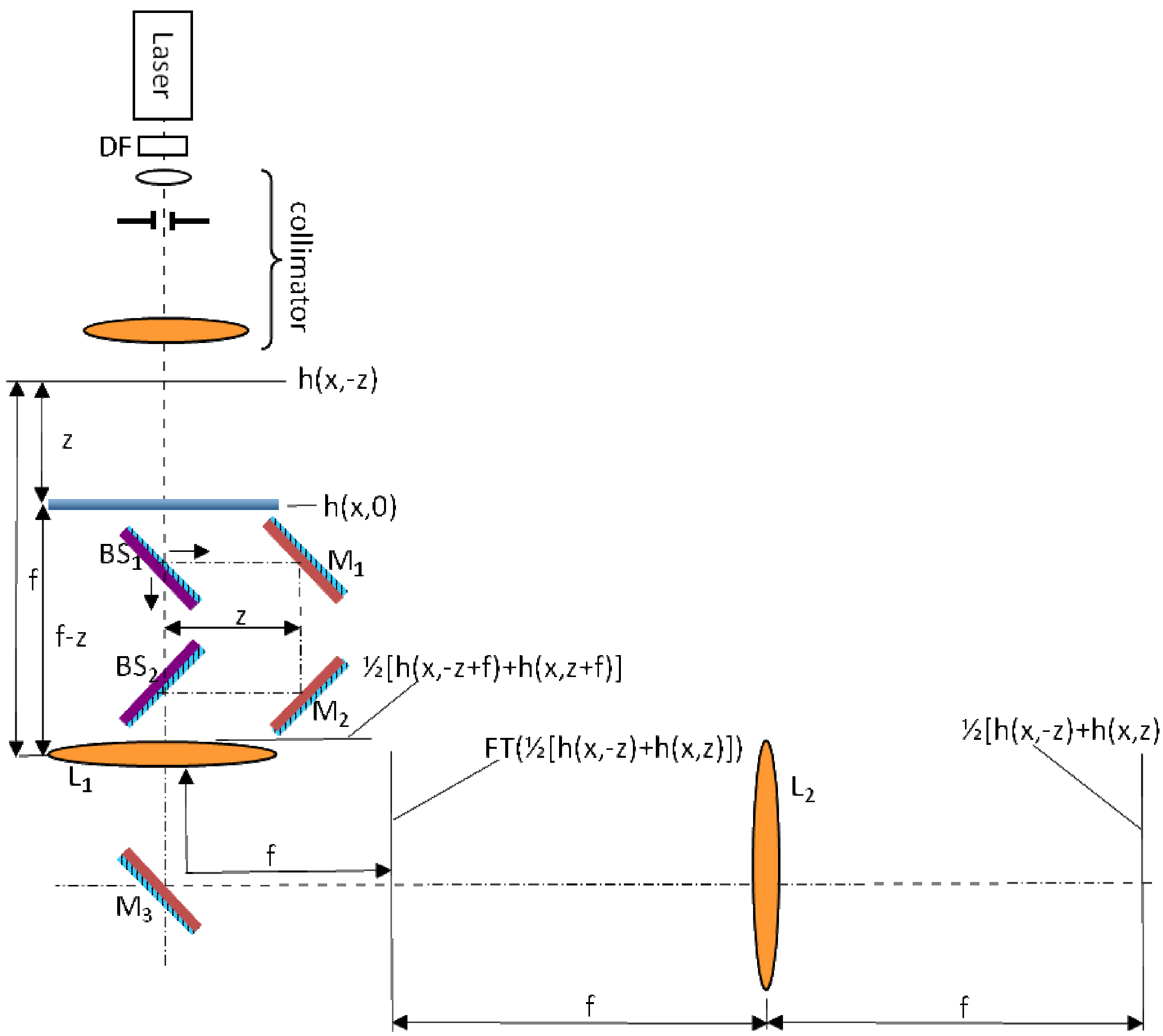

FIGURE 1: A setup to implement the Cosine Fresnel transform optically: a 4-f setup enables reobserving the field $h(x,-z)$ at the back focal plane of the second lens. A beam-splitter BS_1 replicates the field $h(x,0)$ into two replicas. One of the replicas covers an additional distance of $2z$ with respect to the other replica. The two lenses of the 4-f setup may be placed before the creation of replicas as in Figure 2. To obtain the Sine Fresnel transform, we should insert a plate to cause a delay of $\lambda/2$ in one of the arms between the two beam splitters BS_1 and BS_2 as in Figure 2.

In our previous formulas, we considered the space variable *x*, but the analysis is valid for the time variable *t*. It is worth noting that the Fresnel transform is also used to formulate chromatic dispersion in single mode fiber where instead of using the space variable *x*, the time variable t is used [9, 10]. This similarity also let to the Talbot effect in both space and time domain [10,11].

The fractional Fourier transform is expressed as follows for a fractional parameter α [5,6]:

$$h_S(x,z) = \int_{-\infty}^{+\infty} h(x_1,0)\sin\left(\frac{\pi}{\lambda z}(x-x_1)^2\right)dx_1 \qquad (13)$$

with $\csc(\alpha) = 1/\sin(\alpha)$. We see the similarity between Equations (3) and (11).

4. OPTICAL IMPLEMENTATION

The Fresnel transform is implemented naturally by diffraction in the Fresnel zone as well as in the Fraunhofer zone. To optically implement the Sine and Cosine Fresnel transforms, we propose two similar optical setups as depicted in Figures 1 and 2. As will be explained, each of the setups may be used for either transform by adding or omitting a transparent plate to cause a delay of λ/2.

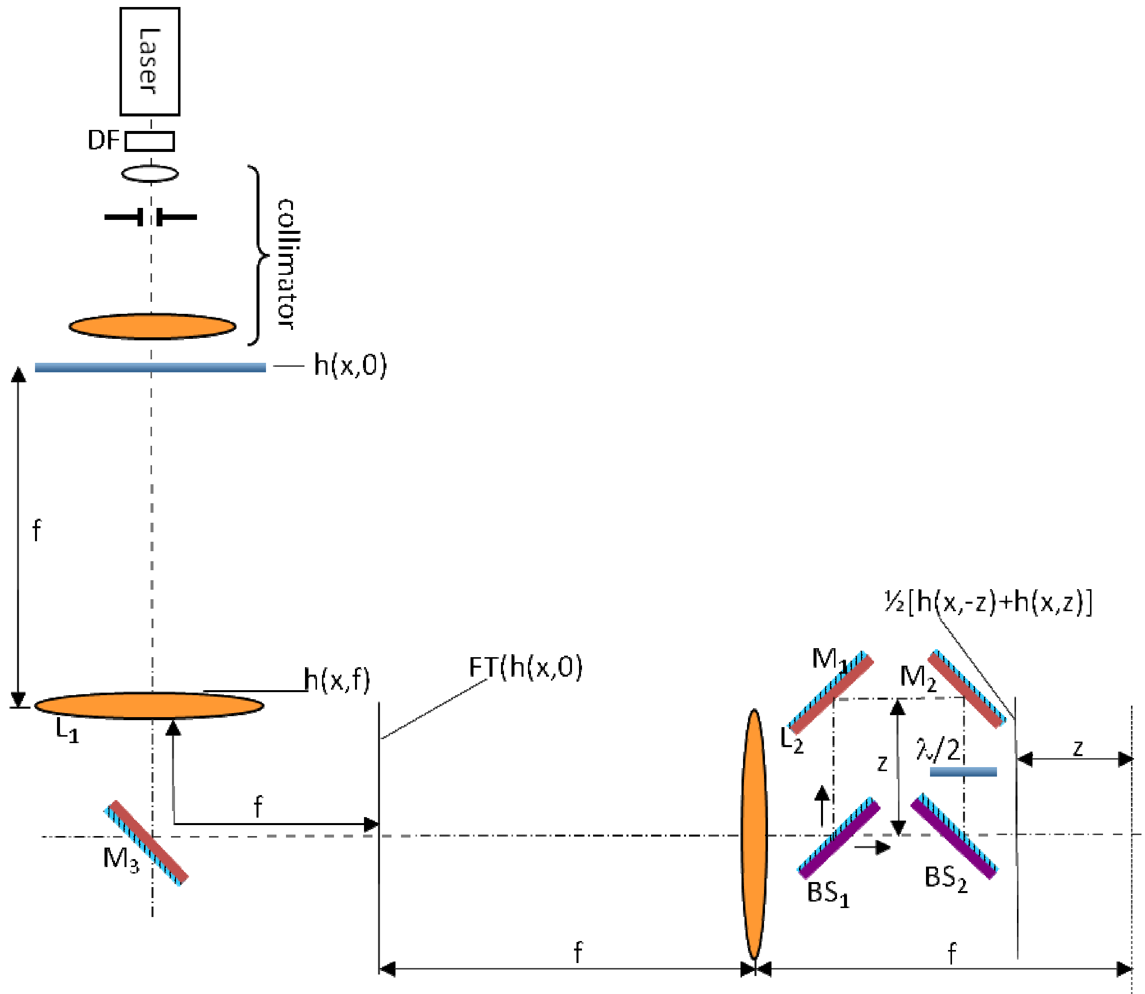

FIGURE 2: An optical setup to implement the Sine Fresnel transform that is identical to Figure 1, except that the two lenses of the 4-f setup is placed before the creation of replicas. Moreover, a plate is inserted to cause a delay of λ/2 in one of the arms between the two beam splitters BS1 and BS2. To obtain the Cosine Fresnel transform, the plate should be omitted.

The idea of both setups is the following: a 4-f setup enables reobserving the initial field at the back focal plane of the second lens. As depicted in Figure 1, we replicate the intermediate diffraction field by means of a beam-splitter BS_1. One of the replicas will cover an additional distance of 2*z* with respect to the other replica. If one replica goes through both converging lenses and covered the distance 4f-z, then it will have the same expression as the field which propagates backwards by a distance z. Thus we obtain ½ *h*(*x*,-*z*). The other replica covers the distance 4f-z+2z=4f+z and is then expressed by ½ h(x,z). In reality, we observe, at the output of the 4-f setup, the mirrored version of a given input field *g*(*x*,0), namely *g*(-*x*,0). This detail is ignored for the sake of simplicity. Thus in the back focal plane of the second lens, we observe ½ [*h*(*x*,-*z*)+*h*(*x*,*z*)], which is the Cosine Fresnel transform of *h*(*x*) as expressed by Equation (5). The expressions of the intermediate fields are given in Figure 1. It is worth noting that the distance *z* should not be so big that mutual coherence between the two light beams, superimposed by means of the second beam splitter BS_2, is lost.

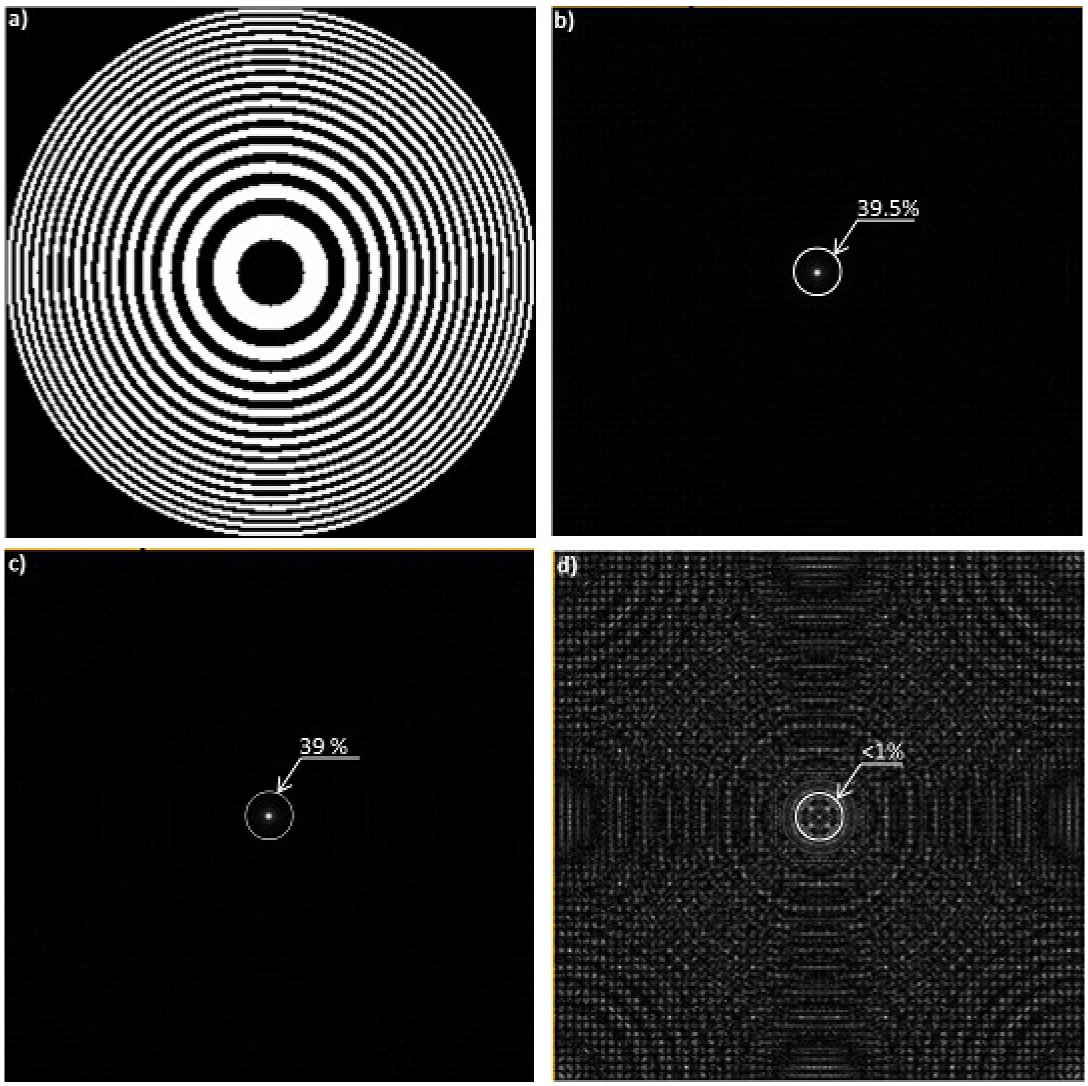

FIGURE 3: Simulation results: a) diffraction object: binary phase only Fresnel zone plate, b) The diffraction pattern at the back focal plane of this binary lens, 39.5% of the light is focused in a point, c) Sine Fresnel transform of the binary lens observed in its back focal plane, 39% of the light is focused in a point, d) Cosine Fresnel transform of the same lens observed in its back focal plane.

Figure 2 presents a setup implementing the Sine Fresnel transform. The setup is identical to that of Figure 1, except that the two lenses of the 4-f setup are placed before the creation of replicas. Moreover, a plate is inserted to cause a delay of $\lambda/2$ in one of the arms between the two beam splitters. To obtain the Cosine Fresnel transform, the plate should be omitted. Similarly, to obtain the Sine Fresnel transform based on the setup of Figure 1, we should insert a plate to cause a delay of $\lambda/2$ in one of the arms between the two beam splitters as in Figure 2.

5. EXPERIMENTAL ILLUSTRATIONS

The sine and cosine functions are two identical trigonometric functions except of a phase shift of $\pi/2$. Thus, we are attempted to state that the Sine and Cosine Fresnel transforms show very similar, not to say identical, behaviors. The illustrations presented in this section are conceived to explore whether the Sine and Cosine transforms show some distinct behaviors.

Let us consider a binary lens, namely a Fresnel zone plate [1] having two phase levels 0 and □ (Fig. 3a). If the first radius of the Fresnel zone plate is r_1, then its focal length is: $f = r_1^2 / \lambda$. Figure 3b points out the diffraction pattern in the back focal plane of the binary lens. Thus Figure 3b presents the result of the Fresnel transform applied at $z=f$. Theoretically, light focused into the focal point could not exceed η=sinc(1/2) ≅40.53% of the light transmitted by the lens since this diffractive element is binary [12]. We notice that sinc(α) = sin($\pi\alpha$)/($\pi\alpha$).

Figures 3c and 3d present the results of the Sine and Cosine transforms respectively, observed in the back focal plane of the binary lens. Figure 3c shows that most of the light, incident to the binary lens, is focused into the focal point with an efficiency of 39%, which is slightly smaller than 40.53%. However, Figure 3d does not show the same behavior. Only a very small part of the incident light (<1%) is focused into the central point of the focal plane. It means that the Sine Fresnel transform shows the converging behavior of the wavefront just behind a binary lens (real distribution), whereas the Cosine Fresnel transform does not.

6. DISCUSSION

To explain the simulation results of Figures 3c and 3d, let us look at the kernels of the Sine and Cosine Fresnel transforms which are expressed by Equations (6) and (5) respectively. The kernel of the Cosine Fresnel transform $f_C(x,z) = \cos\left(\pi x^2 / \lambda z\right)$ is the real part of the Fresnel kernel $f_k(x,z) = \exp\left(j\pi x^2 / \lambda z\right)$, which is nothing but the transmittance of a diverging wavefront $f_k(x,-z) = \exp\left(-j\pi x^2 / \lambda z\right)$ (Fig. 4). Besides, the Cosine Fresnel kernel $f_C(x,z)$ is also the real part of the converging wavefront (Fig. 4), yielding: $f_C(x,-z)=f_C(x,z)$. Thus, the Cosine Fresnel kernel is neutral in terms of convergence or divergence.

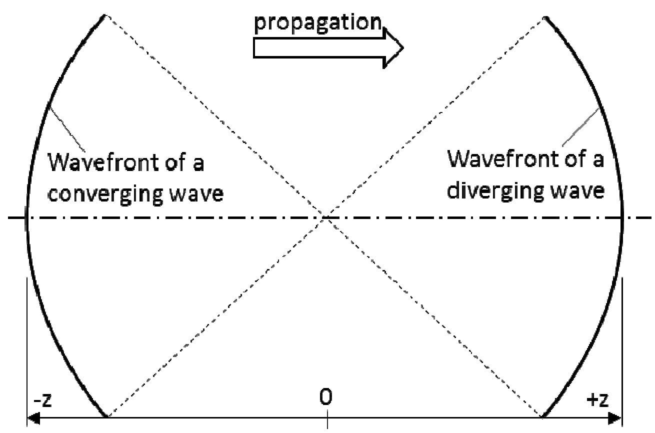

FIGURE 4: Converging and Diverging Wavefronts.

In contrast to the Cosine Fresnel kernel, the Sine Fresnel kernel is sensitive to the direction of propagation (backwards or forwards): $f_S(x,-z)=-f_S(x,z)$. Thus, Sine Fresnel transform is sensitive to the converging or diverging behavior of the wave, represented in our case by a binary real distribution. This explains the observations on Figures 3c and 3d.

For a given initial input distribution $h(x,0)$, that is real, One can obtain the Cosine or Sine Fesnel transform by extracting the real or imaginary part of the Fresnel transform. Since until now there is no receiver that is sensitive to phase, extracting the real or imaginary part of a complex field is not straightforward. It requires a complicated system such as digital or analog holography. In this case, a hologram is required and the Cosine Fesnel transform is general obtained in two steps: recording hologram then reading it. We propose a less complicated system where you only need to place your object in the optical setup and observe the result on the Camera placed in the output plane.

7. CONCLUSION

In conclusion, two novel transforms called Sine and Cosine Fresnel Transforms were advanced. Moreover, two variants of their optical implementation are presented. The advantage of optics is the high speed of processing of digital signals. These two transforms are very useful in both electronics-based (including DSP and FPGA) or optics-based signal processing. In contrast to the Fresnel transform, both Sine and Cosine Fresnel transforms have the advantage of providing a real result for real input distribution. Except of this difference, all three transforms have similar properties and therefore the same domain of application. Since the Fresnel transform is also the optical version of the fractional Fourier transform, both Fresnel transform and the two proposed transforms, namely the Sine and Cosine Fresnel Transform, show the same properties as the fractional Fourier transform. In that sense, the Sine and Cosine Fresnel transforms may be respectively regarded as the fractional Sine and Cosine transforms, which are more general than the Sine and Cosine transforms. The Sine and Cosine Fresnel transforms may be used in all applications where the Sine and Cosine transforms are applied. However, the former transforms offers an additional degree of freedom which is the propagation distance z.

Although derived from the same transform, namely the Fresnel transform, and the only mathematical difference lies in using the cosine or the sine function, the Sine and Cosine Fresnel transforms show some distinct behaviors. The Sine Fresnel transform is sensitive to the converging or diverging behavior of the wave, represented by a binary real distribution, whereas the Cosine Fresnel transform is neutral in that sense.

As mentioned above, in a future work will explore properties of the Sine and Cosine Fresnel transform including self-imaging related properties. The limitations of the optical implementation will be also explored, and especially: limits of z, tolerance interval of z, tolerance interval of the $\lambda/2$ plate.

We will also explore how to use these two transforms in compression, cryptography and steganography, with both variants: fully digital approach or digital and optical approach.

8. REFERENCES

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