Algorithm to Generate Wavelet Transform from an Orthogonal Transform

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Abstract

This paper proposes algorithm to generate discrete wavelet transform from any orthogonal transform. The wavelet analysis procedure is to adopt a wavelet prototype function, called an *analyzing wave* or *mother wave*. Other wavelets are produced by translation and contraction of the mother wave. By contraction and translation infinite set of functions can be generated. This set of functions must be orthogonal and this condition qualifies a transform to be a wavelet transform. Thus there are only few functions which satisfy this condition of orthogonality. To simplify this situation, this paper proposes a generalized algorithm to generate discrete wavelet transform from any orthogonal transform. For an NxN orthogonal transform matrix T, element of each row of T is repeated N times to generate N Mother waves. Thus rows of original transform matrix become wavelets. As an example we have illustrated the procedure of generating Walsh Wavelet from Walsh transform. Since data compression is one of the best applications of wavelets, we have implemented image compression using Walsh as well as Walsh Wavelet. Our experimental results show that performance of image compression technique using Walsh Wavelet is much better than that of standard Walsh transform. More over image reconstructed from Walsh transform has some blocking artifact, which is not present in the image reconstructed from Walsh Wavelet. Similarly image compression using DCT and DCT Wavelet has been implemented. Again the results of DCT Wavelet have been proved to perform better than normal DCT.

Keywords: Wavelet Transform, Walsh Wavelet, DCT Wavelet, Image compression.

1. INTRODUCTION

In the past few years, the study of wavelets and the exploration of the principles governing their behavior have brought about sweeping changes in the disciplines of pure and applied mathematics and sciences. One of the most significant developments is the realization that, in addition to the canonical tool of representing a function by its Fourier series, there is a different representation more adapted to certain problems in data compression, noise removal, pattern classification and fast scientific computations [1, 2]. Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes [1-4]. Fourier analysis is a global scheme. While we can analyze the frequencies that make up a signal, the local properties of the signal cannot be easily detected from the Fourier coefficients. To overcome this difficulty STFT (Short Time Fourier Transform) [5] was introduced. However it gives only local properties at the cost of global properties. Wavelets overcome this shortcoming of Fourier analysis [6, 7] as well as STFT. They provide a way to do a time-frequency analysis simultaneously. Another advantage of wavelet is that they process data at different *scales* or *resolutions.* If we look at a signal with a large "window," we would notice gross features. Similarly, if we look at a signal with a small "window," we would notice local features. The result in wavelet analysis is to see both the forest *and* the trees [1, 3]. The wavelet analysis procedure is to adopt a wavelet prototype function, called an *analyzing wave* or *mother wave*. Other wavelets are produced by translation and contraction of the mother wave. By contraction and translation infinite set of functions can be generated. This set of functions must be orthogonal and this condition qualifies a transform to be a wavelet transform. Thus a transform is qualified to be a wavelet transform if and only if it satisfies the condition of orthogonality. Thus there are only few functions which satisfy these conditions. Principal advantage of wavelet is that they provide time-frequency localization. Temporal analysis is performed with a contracted, high-frequency version of the prototype wavelet, while frequency analysis is performed with a dilated, low-frequency version of the same wavelet. The exact shape of the mother wave strongly affects the accuracy and compression properties of the approximation. Because the original signal or function can be represented in terms of a wavelet expansion (using coefficients in a linear combination of the wavelet functions), data operations can be performed using just the corresponding wavelet coefficients. And if you further choose the best wavelets adapted to your data, or truncate the coefficients below a threshold, your data is sparsely represented. This sparse coding makes wavelets an excellent tool in the field of data compression.

This paper presents a generalized algorithm to develop wavelet from any orthogonal transform. Purpose of this paper is to generate orthogonal discrete wavelet transform from existing orthogonal transform. In [8] Kekre's wavelet transform is generated from Kekre's transform [9]. We propose that the same algorithm can be used to generate wavelets from any orthogonal transform. As an example we have illustrated how Walsh Wavelet can be generated from standard Walsh transform. A necessary condition to generate a discrete wavelet transform from any transform is that the transform from which the wavelet is build must be orthogonal. For example Walsh, DCT, DST, Slant etc. are orthogonal transforms so we can generate Walsh Wavelet, DCT Wavelet, DST Wavelet, Slant Wavelet respectively. The paper is organized as follows. Section 2 discusses a generalized algorithm to generate Wavelet from an orthogonal transform; Section 3 illustrates generation of Walsh Wavelet from Walsh transform. Section 4 discusses properties of Walsh Wavelet. In Section 5 we have explored image compression technique using Walsh and Walsh Wavelet. In Section 6, experimental results of image compression using Walsh and Walsh Wavelet as well as DCT and DCT Wavelet are discussed. Finally in Section 7 the paper is concluded.

2. GENERATING WAVELET FROM ANY ORTHOGONAL TRANSFORM

From any NxN orthogonal transform T, we can generate wavelet transform matrix of size $N^2 x N^2$. For example, from orthogonal transform T of size 5x5, we can generate corresponding Wavelet transform matrix of size 25x25. In general M x M Wavelet transform matrix can be generated from

NxN orthogonal basic transform T such that $M = N^2$. Consider any orthogonal transform T of size NxN shown in FIGURE 1.

T_{11}	T_{12}	T_{13}	…	$T_{1\,(N-1)}$	T_{1N}
T_{21}	T_{22}	T_{23}	…	$T_{2\,(N-1)}$	T_{2N}
T_{31}	T_{32}	T_{33}	…	$T_{3\,(N-1)}$	T_{3N}
$\vdots$	$\vdots$	$\vdots$	…	$\vdots$	$\vdots$
T_{N1}	T_{N2}	T_{N3}	…	$T_{N\,(N-1)}$	T_{NN}

FIGURE 1: NxN orthogonal transform

1st Column of T repeated N times

2nd Column of T repeated N times

Nth Column of T repeated N times

T_{11}	T_{11}	…	T_{11}	T_{12}	T_{12}	…	T_{12}	…	T_{1N}	T_{1N}	…	T_{1N}
T_{21}	T_{21}	…	T_{21}	T_{22}	T_{22}	…	T_{22}	…	T_{2N}	T_{2N}	…	T_{2N}
T_{31}	T_{31}	…	T_{31}	T_{32}	T_{32}	…	T_{32}	…	T_{3N}	T_{3N}	…	T_{3N}
$\vdots$	$\vdots$	… …	$\vdots$	$\vdots$	$\vdots$	… …	$\vdots$	…	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T_{N1}	T_{N1}	…	T_{N1}	T_{N2}	T_{N2}	…	T_{N2}	…	T_{NN}	T_{NN}		T_{NN}
T_{21}	T_{22}	…	T_{2N}	0	0	…	0	…	0	0	0	0
0	0	…	0	T_{21}	T_{22}	…	T_{2N}	…	0	0	0	0
	$\vdots$							$\vdots$				
0	0	…	0	0	0	…	0	…	T_{21}	T_{22}	…	T_{2N}
T_{31}	T_{32}	…	T_{33}	0	0	…	0	…	0	0	…	0
0	0	…	0	T_{31}	T_{32}	…	T_{33}	…	0	0	…	0
	$\vdots$							$\vdots$				
0	0	…	0	0	0	…	0	…	T_{31}	T_{32}	…	T_{33}
		… … …				… … …		… … …			… … …	
T_{N1}	T_{N2}	…	T_{NN}	0	0	…	0		0	0	…	0
0	0	…	0	T_{N1}	T_{N2}	…	T_{NN}		0	0	…	0
0	0	…	0	0	0	…	0		T_{N1}	T_{N2}	…	T_{NN}

FIGURE 2: MxM Wavelet transform matrix ($M=N^2$) generated from orthogonal transform T of size NxN.

FIGURE 2 shows MxM Wavelet transform matrix ($M = N^2$) generated from NxN orthogonal transform T. First N number of rows of Wavelet transform matrix is generated by repeating every column of transform T, N times. To generate next (N+1) to 2N rows, second row of transform T is translated. To generate next (2N+1) to 3N rows, third row of transform T is translated. Like wise to generate last ((N-1)N + 1) to N^2 rows, N^{th} row of transform T is used. Note that by repeating every column of the basic transform N times we are generating Mother wave. Other wavelets are generated by contraction and translation of the mother wave.

3. GENERATING WALSH WAVELET FORM WALSH TRANSFORM

In [8] Kekre's Wavelet transform is generated from Kekre's Transform [9]. From NxN Kekre's transform matrix we can generate 2Nx2N, 3Nx3N,…,N^2xN^2 Kekre's Wavelet transform. Similar technique can be applied to any orthogonal transform of size NxN to generate wavelet of size N^2xN^2. Since Walsh is an orthogonal transform, we can generate Wavelet from Walsh transform. Consider 4x4 Walsh transform matrix shown in FIGURE 3. Procedure of generating 16x16 Walsh Wavelet from 4x4 Walsh transform is illustrated in FIGURE 4.

1	1	1	1
1	1	-1	-1
1	-1	-1	1
1	-1	1	-1

FIGURE 3: 4 x 4 Walsh Transform matrix W_4

	1st column of W_4 Repeated N=4 times				2nd column of W_4 repeated N=4 times				3rd column of W_4 repeated N=4 times				4th column of W_4 repeated N=4 times			
1 to 4 rows	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1
	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
5 to 8 rows	1	1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	1	-1	-1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	1	-1	-1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1	-1	-1
9 to 12 rows	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	-1	1
13 to 16 rows	1	-1	1	-1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	1	-1	1	-1	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	-1	1	-1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	-1	1	-1

FIGURE 4: 16x16 Walsh Wavelet generated from 4x4 Walsh transform

4. PROPERTIES OF WALSH WAVELET

4.1 Orthogonal

The transform matrix M is said to be orthogonal if the following condition is satisfied.

$$[M][M]^T = [D]$$

Where D is the diagonal matrix. $N^2 \times N^2$ Wavelet generated from any orthogonal transform of size NxN satisfies this property and hence it is orthogonal.

4.2 Non Involutional

An involutionary function is a function that is its own inverse [7]. So involutionar transform is a transform which is inverse transform of itself. Walsh Wavelet is non involutional transform.

4.3 Transform on Vector

The Walsh Wavelet transform (WLT) on a column vector f is given by

$$F = [WLT]\, f$$

And inverse is given by

$$f = [WLT]^T\ [D]^{-1}\ F$$

4.4 Transform on 2D Image

Walsh Wavelet transform on 2D image [f] is given by

$[F] = [WLT]\ [f]\ [WLT]^T$

Obtaining Inverse:

Calculate Diagonal matrix D (FIGURE 5) as,

$[D] = [WLT]\ [WLT]^T$

D1	0	0	0	0	0
0	D2	0	0	0	0
0	0	D3	0	0	0
0	0	0	…	0	0
0	0	0	0	…	0
0	0	0	0	0	DN

FIGURE 5: Diagonal matrix D

Inverse is calculated as

$[f] = [WLT]^T\ [\ F_{ij} / D_{ij}\ ]\ [WLT]$

Where $D_{ij} = D_i * D_j$; $1 \le i \le N$ and $1 \le j \le N$

5. IMAGE COMPRESSION USING WALSH AND WALSH WAVELET

Image compression is the process of converting image files into smaller files for efficiency of storage and transmission [10-13]. Thus a compression is the process of representing data in compact form. In this section, we discuss the algorithm to compress an image using Walsh transform and Walsh Wavelet.

5.1 Image Compression using Walsh:

Step 1: Walsh transform is first applied on a full image. We get the transformed image.

Step 2: The transformed image is then divided into 64 equal non-overlapping blocks.

Step 3: Energy of each block is computed as summation of square of the coefficients within that block.

Step 4: Sort all 64 blocks in ascending order according to their energies. Thus the first block in the sorted list is the lowest energy block and the last block in the sorted list is the highest energy block.

Step 5: Input number of blocks to be compressed say M.

Step 6: Compress first M blocks (of the transformed image) from the sorted list. Compressing an image is nothing but making coefficients of the selected block zero.

Step 7: Apply Inverse Walsh transform to reconstruct the image.

5.2 Image Compression using Walsh Wavelet

Step 1: Walsh Wavelet Transform (WLT) is first applied on full image f of size NxN. The resultant image F is,

$$F = [WLT]\,[f]\,[WLT]^T$$

Step 2: The diagonal matrix D is computed as $D = [WLT]\,[WLT]^T$

Step 3 : Compute G = [Fij /Dij] i.e. every element of transformed image F is divided by corresponding Dij value.

$$\text{Where } D_{ij} = D_i * D_j \quad ; 1 \le i \le N \text{ and } 1 \le j \le N$$

Step 4: Calculate energy of each element of G as,

$$E_{ij} = [F_{ij} / D_{ij}]^2 \times D_{ij}$$

Step 5: Divide the whole G image into 64 equal non-overlapping blocks. Compute energy of each block as summation of energy of each element within the block.

Step 6: Calculate percentage energy of each block.

Step 7: Sort the blocks in ascending order of their percent energy.

Step 8: Input number of blocks to be compressed say M.

Step 9: Compress first M number of blocks from the sorted list (Make all coefficients of the block zero)

Step 10: Apply inverse Walsh Wavelet transform to reconstruct the image.

$$\text{Reconstructed\_f} = [WLT]^T\,[G]\,[WLT]$$

6. RESULTS AND DISCUSSION

Algorithms discussed in previous section were implemented on a bitmap gray image of size 256x256. Image was compressed using both Walsh as well as Walsh Wavelet transform as explained in section 5. 256x256 Walsh Wavelet was generated from 16x16 Walsh transform and applied on to the image. TABLE1 compares the results of Data compression using Walsh and Walsh Wavelet.

Image	Compression Ratio	No of Blocks compressed	Walsh		Walsh Wavelet		MSE Reduction factor	Increased PSNR
			PSNR	MSE	PSNR	MSE		
Horse	0.625	40	27.89	105.61	35.37	18.86	5.599	7.48
	0.75	48	27.46	116.61	32.7	34.9	3.32	5.24
	0.82	53	27.14	125.48	29.12	79.62	1.57	1.98
Temple	0.625	40	24.72	218.85	32.69	34.93	6.26	7.97
	0.75	48	24.46	232.41	29.09	80.16	2.89	4.63
	0.82	53	24.21	246.26	27.07	127.42	1.93	2.86
Rose	0.625	40	33.83	26.89	43.08	3.19	8.42	9.25
	0.75	48	33.45	29.35	38.47	9.23	3.17	5.02
	0.82	53	33.14	31.54	35.42	18.66	1.69	2.28
Puppy	0.625	40	30.04	64.34	34.2	24.68	2.60	4.16
	0.75	48	29.56	71.83	32.33	38	1.89	2.77
	0.82	53	29.19	78.17	30.97	51.94	1.50	1.78

TABLE 1: Results of image compression using Walsh and Walsh Wavelet.

Our experimental results clearly show that performance of Walsh Wavelet is far better than Walsh transform for compression. Compression using Walsh Wavelet achieves better PSNR and MSE values for all type of images. On an average MSE is reduced by factor of 5 to 8, 2 to 3 and 1.5 to 2.0 for compression ratio 0.625, 0.75 and 0.82 respectively. PSNR is increased by 7 to 9, 4 to 5 and 2 to 3 dB for compression ratio 0.625, 0.75 and 0.82 respectively.

FIGURE 6 shows the original and reconstructed image using Walsh and Walsh Wavelet.

FIGURE 6 (a): Original Image

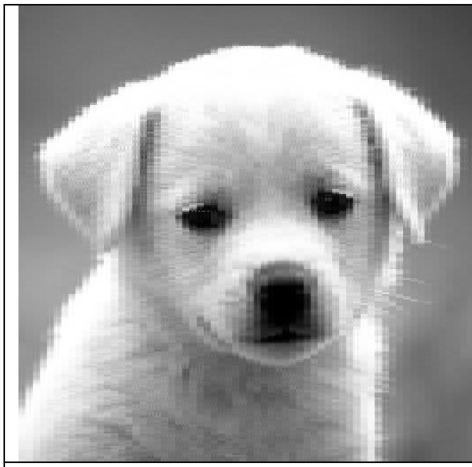	
Reconstructed Image (Walsh)	Reconstructed Image (Walsh Wavelet)

FIGURE 6 (b): Reconstructed Image (No. Of Blocks Compressed = 40)

	
Reconstructed Image (Walsh)	Reconstructed Image (Walsh Wavelet)

FIGURE 6 (c): Reconstructed Image (No. Of Blocks Compressed = 48)

	
Reconstructed Image (Walsh)	Reconstructed Image (Walsh Wavelet)

FIGURE 6 (d): Reconstructed Image (No. Of Blocks Compressed = 53)

Our results show that reconstructed image using Walsh has blocking artifact, where as image reconstructed from Walsh Wavelet is free from this type of blocking artifact.

Similarly we have implemented image compression using DCT and DCT Wavelet. Again DCT Wavelet of size 256x256 is generated from 16x16 DCT. TABLE 2 summarizes the results of image compression technique using DCT and DCT Wavelet.

Image	Compression Ratio	No of Blocks compressed	DCT		DCT Wavelet		MSE Reduction factor	Increased PSNR
			PSNR	MSE	PSNR	MSE		
Horse	0.625	40	31.21	49.2	40.66	5.58	8.81	9.45
	0.75	48	31.19	49.33	36.68	13.95	3.53	5.49
	0.82	53	31.07	50.78	33.9	26.44	1.92	2.83
Temple	0.625	40	27.74	109.32	38.02	10.24	10.67	10.28
	0.75	48	27.73	109.58	33.67	28.52	3.84	5.94
	0.82	53	27.65	111.56	30.66	55.85	1.99	3.01
Rose	0.625	40	38.83	8.51	52.23	0.38	22.39	13.4
	0.75	48	38.83	8.51	47.57	1.13	7.53	8.74
	0.82	53	38.75	8.66	42.24	3.87	2.23	3.49
Puppy	0.625	40	33.09	31.85	35.49	18.36	1.73	2.4
	0.75	48	32.79	34.18	33.86	26.69	1.28	1.07
	0.82	53	32.37	37.64	32.69	34.96	1.07	30.32

TABLE 2: Results of image compression using DCT and DCT Wavelet

Again the image compression using DCT Wavelet is proved to be superior to image compression using DCT. FIGURE 7 (a) compares the MSE values obtained from Walsh, Walsh Wavelet, DCT and DCT Wavelet for four different gray images for 62.5% compression that is 40 out of 64 blocks are compressed.

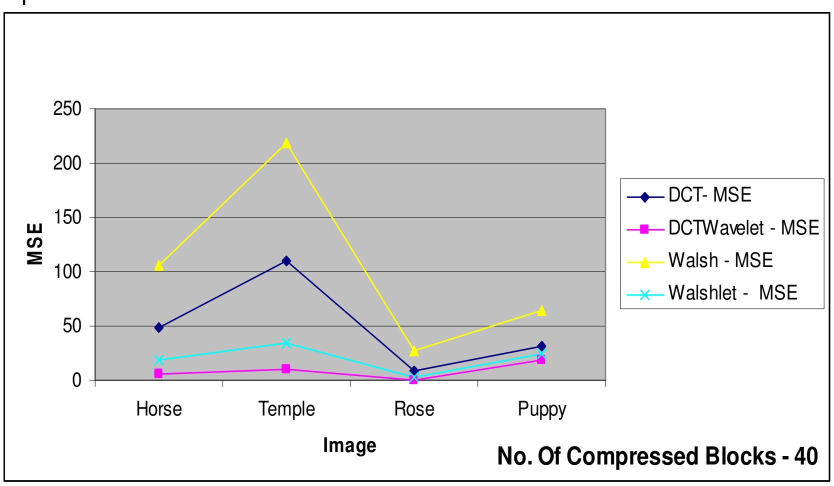

FIGURE 7 (a): Comparison of Data compression technique using DCT, DCT Wavelet, Walsh and Walsh Wavelet with respect to MSE (For Compression ratio 0.625)

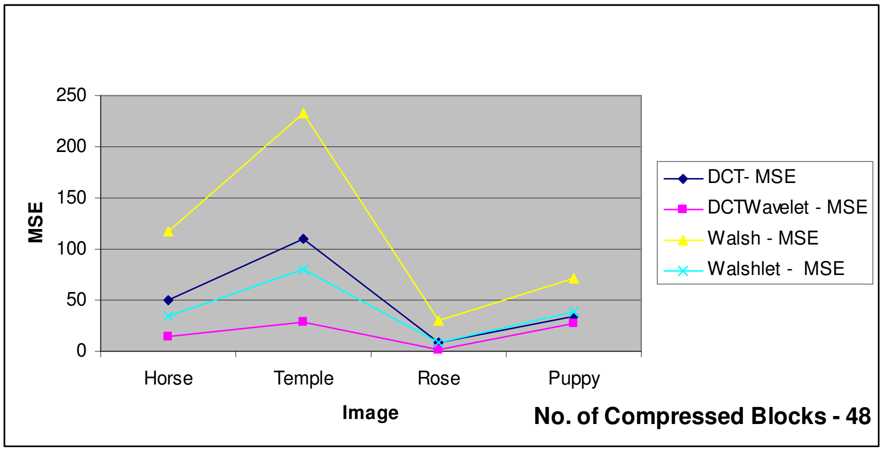

FIGURE 7 (b): Comparison of Data compression technique using DCT, DCT Wavelet, Walsh and Walsh Wavelet with respect to MSE (For Compression ratio 0.75)

From FIGURE 7 it is clear that MSE value between original image and reconstructed image of data compression technique using DCT Wavelet and Walsh Wavelet is much lower than that of using DCT and Walsh.

7. CONCLUSION

The wavelet analysis procedure is to adopt a wavelet prototype function, called an *analyzing wave* or *mother wave*. Other wavelets are produced by translation and contraction of the mother wave. A transform thus formed qualifies to be a wavelet transform if and only if it satisfies the condition of orthogonality. Thus there are only few functions which satisfy these conditions. To simplify this situation, this paper proposes a generalized algorithm to generate discrete wavelet transform from any orthogonal transform. For an NxN orthogonal transform matrix T, element of each row of T is repeated N times to generate N Mother waves. Thus rows of original transform matrix become wavelets. As an example we have illustrated how Walsh Wavelet is generated from Walsh transform. Image compression using Walsh Wavelet has been implemented and the results are compared with image compression using Walsh transform. Compression using Walsh Wavelet achieves much better PSNR and MSE values for all type of images. On an average MSE is reduced by factor of 5 to 8, 2 to 3 and 1.5 to 2.0 for compression ratio 0.625, 0.75 and 0.82 respectively. PSNR is increased by 7 to 9, 4 to 5 and 2 to 3 dB for compression ratio 0.625, 0.75 and 0.82 respectively. Results of compression have been observed for DCT and DCT Wavelet as well. Again the image compression using DCT Wavelet is proved to be superior to image compression using DCT. Moreover the reconstructed image from Walsh Wavelet is free from blocking artifact which appears in reconstructed image from Walsh Transform.

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