Design of Low-Pass Digital Differentiators Based on B-splines

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Abstract

This paper describes a new method for designing low-pass differentiators that could be widely suitable for low-frequency signals with different sampling rates. The method is based on the differential property of convolution and the derivatives of B-spline bias functions. The first order differentiator is just constructed by the first derivative of the B-spline of degree 5 or 4. A high (>2) order low-pass differentiator is constructed by cascading two low order differentiators, of which the coefficients are obtained from the *n*th derivative of a B-spline of degree *n*+2 expanded by factor *a*. In this paper, the properties of the proposed differentiators were presented. In addition, we gave the examples of designing the first to sixth order differentiators, and several simulations, including the effects of the factor *a* on the results and the anti-noise capability of the proposed differentiators. These properties analysis and simulations indicate that the proposed differentiator can be applied to a wide range of low-frequency signals, and the trade-off between noise-reduction and signal preservation can be made by selecting the maximum allowable value of *a*.

Keywords: Low-pass Differentiator, B-spline, Finite-impulse Response (FIR), Digital Filters.

1. INTRODUCTION

Digital differentiators (DDs) have been applied in several areas, such as radar, sonar, communication systems and signal processing system [1-3]. In particularly, low-pass high-order DDs are utilized in biological and electrochemical signal processing etc [4, 5]. Since the signal values are known on discrete points because of sampling operation, difference approximation is usually used to design DDs [6]. However, differentiation could amplify the noises contaminating the signal, especially the high-frequency noises [4, 5, 7]. And the signals we need to study, such

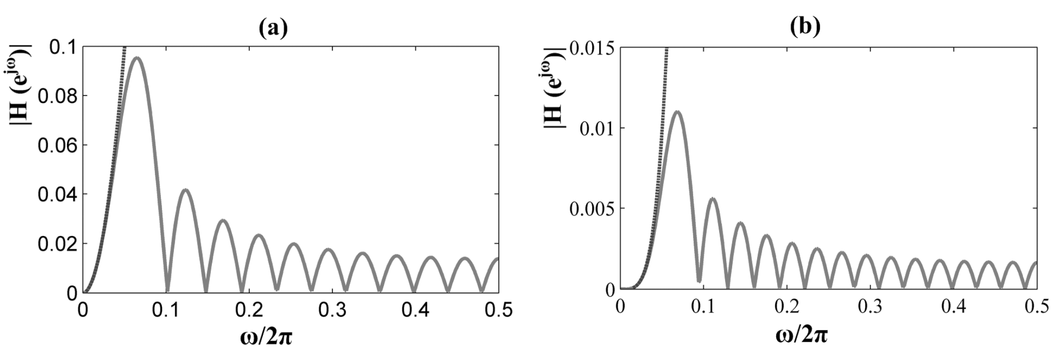

FIGURE 1: The frequency responses of the Savitzky-Golay digital differentiators (SGDDs) (—) and the corresponding ideal differentiators (---). (a) is the frequency response of the 2nd order SGDD by using fitting coefficients of fourth-order polynomials on 25 points, and (b) is the frequency response of the 4th order SGDD by using fitting coefficients of sixth-order polynomials on 35 points.

as biological and electrochemical signals, are mostly at low frequencies. Therefore, low-pass digital differentiators (LPDDs) have been to estimate the derivatives [5, 7, 8].

Many methods have been available for the design of LPDDs. Most of them focus on the first order differentiators [8-10], which cannot directly obtain the high order derivatives of the signals. The Savitzky-Golay digital differentiators (SGDDs) are generally used for smoothing and acquiring low or high order derivatives due to their low-pass characteristic and arbitrary lengths etc [5]. But there are several weak points for SGDDs. One hand, the filter length and the degree of fitting polynomials can be arbitrarily selected, which instead, makes it blind in selections. Although, recent researches have focused on adaptive extension of the SG approach [11-13], they may increase the complexity of the algorithm, and there is still a need for further researches and tests. On the other hand, the frequency responses of SGDDs have several ripples at high frequencies, and the frequency responses of even order SGDDs at $\omega = \pi$ are not zero (Figure 1), which may affect the results of SGDDs filtering the high-frequency noises [5, 14].

In order to meet the low-pass characteristic and apply to different types of signals, we propose a method for designing LPDDs based on B-splines by using the differential property of convolution. B-splines have been widely used in data smooth because of their explicit formulae and Gaussian-like waveforms [15, 16]. Moreover, the derivatives of B-spline bias functions are continuous and easily obtained. Consequently, B-splines have been used to calculate the derivatives of the gray of the image [15, 17]. However, they have not been widely used to obtain the derivatives of sampled signals.

The aim of this study, therefore, was to propose a method for designing any order LPDDs, which were simple, flexible, easy to control and suitable for low-frequency signals with various sampling rates. In this paper, we first introduced the method of the designs of LPDDs. Then, several properties of the proposed LPDDs were summarized, and some computer simulations of various orders LPDDs, acting on the input testing signals produced by a Gaussian function in different ways, were also presented.

2. THEORIES

2.1 Background

Let β_m denote the mth order central B-spline function that can be generated by repeated convolutions of a B-spline of degree 1

$$\beta_{\mathrm{m}}(t) = \beta_{\mathrm{m\text{-}1}}(t) * \beta_1(t) \tag{1}$$

where $\beta_1(t)$ is the indicator function in the interval [-1/2, 1/2], and the derivatives of central B-splines can be obtained in a recursive fashion based on the following property [15, 18]:

$$\frac{d\beta_{\mathrm{m}}(t)}{d\mathrm{t}} = \beta_{\mathrm{m\text{-}1}}(t + \frac{1}{2}) - \beta_{m-1}(t - \frac{1}{2}) \tag{2}$$

If $f(t)$ denotes a continuous signal, and $\beta_m(t/a)$ is the B-spline of degree m expended by factor a, the convolution between $f(t)$ and the nth derivative of $\beta_m(t/a)$ could be written as:

$$\begin{aligned} W(t) &= f(t) * \beta_{\mathrm{m}}^{(n)}(t/a) = f(t) * \frac{d^n}{d(t/a)^n}\beta_m(t/a) \\ &= f(t) * \left[a^n \cdot \frac{d^n}{dt^n}\beta_m(t/a)\right] \\ &= a^n \cdot \int \beta_m(t/a)\mathrm{d}t \cdot \frac{d^n}{dt^n}\left[\frac{f(t) * \beta_m(t/a)}{\int \beta_m(t/a)\mathrm{d}t}\right] \\ &= a^{n+1} \cdot \frac{d^n}{dt^n}\left[f(t) * \beta_m(t/a)/a\right] \end{aligned} \tag{3}$$

which is based on the differential property of convolution. The B-spline functions become more and more Gaussian-like with the degree *m* increasing [15], and therefore the convolution between $f(t)$ and $\beta_m(t/a)$ is really to smooth $f(t)$ by $\beta_m(t/a)$ in (3). Accordingly, $W(t)$ could be taken as the dilation of the *n*th derivative of $f(t)$ smoothed by $\beta_m(t/a)$.

When the signal $f(t)$ is sampled once every T seconds, (3) could be written as:

$$\begin{aligned} W(t) &= \mathrm{f}(t) * \beta_{\mathrm{m}}^{(n)}(t/a) = \int f(\tau) * \beta_m^{(n)}(\frac{t-\tau}{a})d\tau \\ &\approx T \times \sum_i f(iT) \cdot \beta_m^{(n)}\left[(j-i)T/a\right] \end{aligned} \qquad (4)$$

where T is the sampling interval, i represents the sample number, and $jT = t$. Since $\beta_m(kT/a)$ is the discrete representation of $\beta_m(t/a)$, the discrete Fourier transform of the sequence $\{\beta_m(kT/a)\}$ and the corresponding *n*th derivative are respectively defined as $\Phi_m(e^{j\omega})$ and $U_{m,n}(e^{j\omega})$. Figure 2 shows the frequency responses of $\Phi_m(e^{j\omega})$ depending on the values of a and m. When a and m increase, the 3 dB cut-off frequency decreases, and the effect of low-pass filtering tends to be more noticeable. In Appendix A, we prove that the 3 dB cut-off frequency f_c of $\Phi_m(e^{j\omega})$ is independent of sampling frequency when a and m are constants. Moreover, for a given B-spline of degree m, the relationship between f_c and a displays as following (see Appendix B):

$$f_c \cdot a = f_m \qquad (5)$$

where the value of f_m is only determined by the degree of the B-spline bias function. According to (3), we also know that the 3 dB cut-off frequency of differentiators designed by $\beta^{(n)}{}_m(t)$ is actually the 3 dB cut-off frequency f_c of $\Phi_m(e^{j\omega})$.

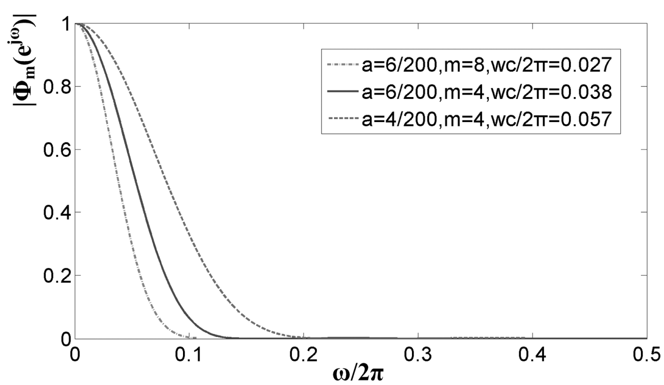

FIGURE 2: The frequency responses of $\Phi_m(e^{j\omega})$ for different values of factor a and degree m when the sampling frequency is 200 Hz.

2.2 The Designs of LPDDs

(3) tells us how to obtain the *n*th derivative of a signal. Obviously, a B-spline of high degree can well smooth the signal $f(t)$ by convolution. However, it may filter some useful information contained in the signal, and need larger computations at the same time. To make the differentiators easy and avoid complicated computations, we design the LPDDs by cascading two low order differentiators.

Usually, the *n*th derivative of a B-spline of degree $n+2$ comprises of piecewise linear polynomials (see Appendix C), which could construct each of the two low order differentiators. Table 1 shows the designs of the 2nd to 6th order differentiators by cascading two low order differentiators, and the first order differentiator is just constructed by the first derivative of the B-spline of degree 5 or 4. In the progress of the algorithm implementation, $f(t)$ is convoluted with the coefficients of the two low order differentiators by using the associative property of convolution as shown in (6).

Although it seems like that the two consecutive convolution operations increase the computational cost, the convolution operations between $f(t)$ and the two low order differentiators avoids calculating the polynomial of high degree in t because the low order differentiator is constructed by piecewise linear polynomials. Moreover, different combinations of several low order differentiators could construct more high order LPDDs.

$$\mathrm{f}(t) * \left[\beta_{m1}^{(n1)}(\frac{t}{a}) * \beta_{m2}^{(n2)}(\frac{t}{a}) \right] = \left[f(t) * \beta_{m1}^{(n1)}(\frac{t}{a}) \right] * \beta_{m2}^{(n2)}(\frac{t}{a}) \qquad (6)$$

If $\{f(iT)\}$ is the input sequence of a single differentiator of degree n, and $\{y[j]\}$ is normalized output, the relationship between $y[j]$ and $f(iT)$ displays in (7).

$$y[j] = (T/a)^{n+1} \cdot \sum_{i=-N}^{N} \beta_m^{(n)}(\frac{iT}{a}) f[(j-i)T] \qquad (7)$$

$$\beta_{m1}^{(n1)}(t/a) * \beta_{m2}^{(n2)}(t/a) = a \cdot \beta_{m1+m2}^{(n1+n2)}(t/a) \qquad (8)$$

where the value of a is usually a multiple of T, and N is the largest integer less than $a{\cdot}m/(2T)$. In addition, (8) is derived by the convolution property. So the 3 dB cut-off frequency of the two cascading low order differentiators is just that of the differentiator constructed by $\beta^{(n1+n2)}_{(m1+m2)}(t)$, and is also equal to that of the filter constructed by the $\beta_{m1+m2}(t)$. By using (5), we get the corresponding equations of a and f_c as shown in Table 1.

The Order of Differentiators	Designs of Differentiators	The Relationship Between *a* and f_c
1	$\beta_4^{(1)}(t/a)$	$f_c \times a = 0.228$
1	$\beta_5^{(1)}(t/a)$	$f_c \times a = 0.204$
2	$\beta_3^{(1)}(t/a) * \beta_3^{(1)}(t/a)$	$f_c \times a = 0.187$
3	$\beta_3^{(1)}(t/a) * \beta_4^{(2)}(t/a)$	$f_c \times a = 0.173$
4	$\beta_4^{(2)}(t/a) * \beta_4^{(2)}(t/a)$	$f_c \times a = 0.162$
5	$\beta_4^{(2)}(t/a) * \beta_5^{(3)}(t/a)$	$f_c \times a = 0.152$
6	$\beta_5^{(3)}(t/a) * \beta_5^{(3)}(t/a)$	$f_c \times a = 0.145$

TABLE 1: The designs of low-pass digital differentiators (LPDDs) and the relationship between the factor a and the 3 dB cut-off frequency f_c of the corresponding B-spline filters.

3. THE PROPOSED DIFFERENTIATORS

3.1 Low-pass Characteristic

Using (7) and (8), we obtain the frequency response of the differentiator of degree n ($n = n1+n2$).

$$H_1(\mathrm{e}^{j\omega}) = T^2 \cdot a^{-2} \cdot \sum_{i=-\infty}^{+\infty} \beta_m^{(1)}(iT/a) e^{-j\omega i} \quad n = 1 \qquad (9)$$

$$\begin{aligned} H_n(\mathrm{e}^{j\omega}) &= H_{n1}(\mathrm{e}^{j\omega}) \cdot H_{n2}(\mathrm{e}^{j\omega}) \\ &= \left[T^{n1+1} \cdot a^{-n1-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m1}^{(n1)}(iT/a) e^{-j\omega i} \right] \cdot \quad n > 1 \\ &\left[T^{n2+1} \cdot a^{-n2-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m2}^{(n2)}(iT/a) e^{-j\omega i} \right] \end{aligned} \qquad (10)$$

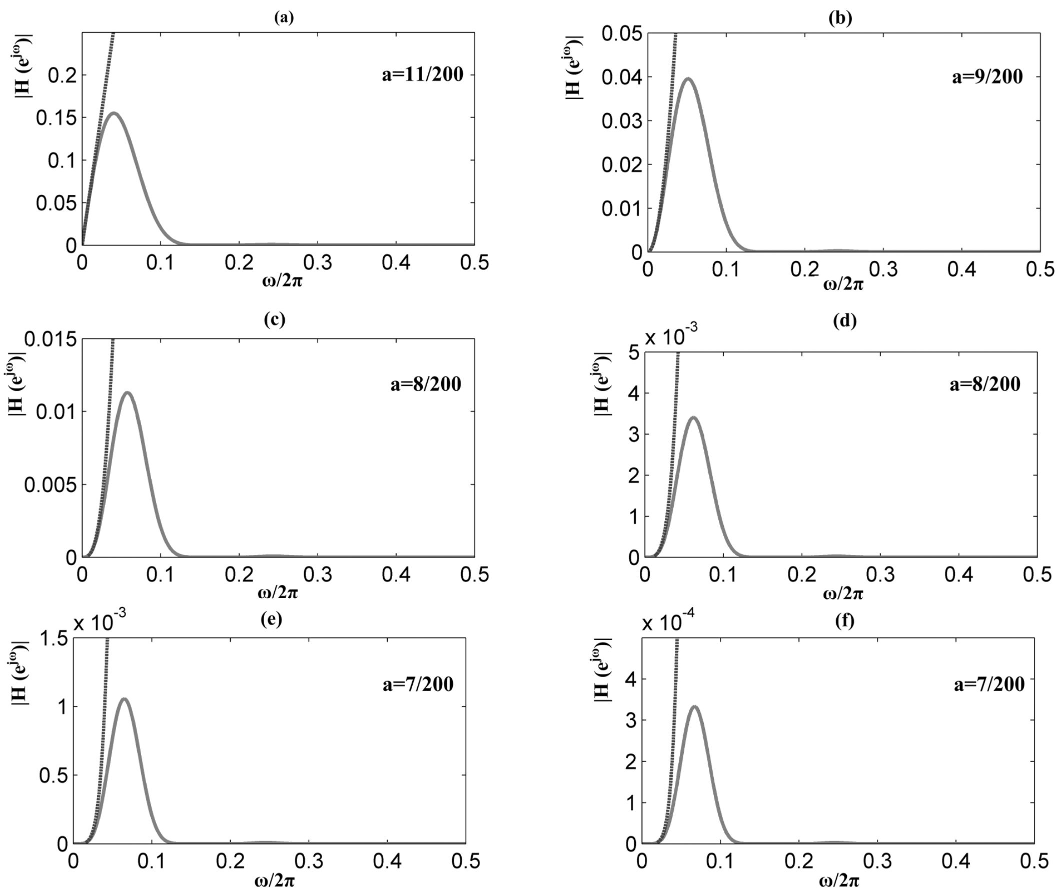

FIGURE 3: The solid lines are the frequency responses of the proposed differentiators of degree 1 (a), degree 2 (b), degree 3 (c), degree 4 (d), degree 5 (e), and degree 6 (f) when the sampling frequency is 200 Hz. The dotted lines are the those of the corresponding ideal differentiators.

As can be seen from Figure 3, the amplitude of $H_n(e^{j\omega})$ is close to the frequency response of the ideal differentiator at low frequencies, and rapidly decays to zero with few ripples, making the differentiator filter high-frequency noises effectively.

3.2 Flexible and Easy to Control

The cut-off frequency is one of the key parameters of a filter. (5) and (8) indicate that the cut-off frequency of a proposed nth order differentiator is only determined by a. Knowing the effective frequency band of the sampled signal, we can use the equation of 3 dB cut-off frequency and a shown in Table 1 to obtain the maximum value of a, which is usually a multiple of T.

3.3 Impulse Response Restriction

If $h_n[i]$ denotes the impulse response sequence of a single differentiator of degree n, using (9) and (10), we derive

$$h_n[i] = (T / a)^{n+1} \cdot \beta_m^{(n1)}[(i-N)T / a] \quad i = 0, 1 \ldots 2N \quad (11)$$

Obviously $\{h_n[i]\}$ is a finite-length sequence. If a nth order differentiator constructed by cascading two low order differentiators, of which degree are respectively $n1$ and $n2$, we can get the impulse response sequence

$$h_{n1+n2}[i] = h_{n1}[i] * h_{n2}[i] \qquad (12)$$

Similarly, $\{h_{n1+n2}[i]\}$ is also a finite-length sequence. Therefore, the proposed differentiators based on B-splines are finite impulse response differentiators. This property is consistent with B-spline filters [15].

3.4 A Low-Complexity Algorithm

According to (6), calculating the derivative of $f(t)$ is just the discrete convolution between $f(iT)$ and $\beta^{(n1)}_{m1}(iT/a)$ without any other filtering algorithms. Moreover, the process of convolutions avoids calculating the polynomial of high degree in t, for $\beta^{(n1)}_{m1}(iT/a)$ comprises of piecewise linear polynomials.

3.5 A Flexible And Easy To Control Frequency Response Flatness At ω = 0

The frequency response of the ideal full-band nth order DD is [9, 14]

$$H_{FBn}(e^{j\omega}) = (j\omega)^n \qquad (13)$$

As illustrated in Appendix D, the frequency response of the proposed differentiator of degree n satisfies the flatness constraints

$$\left|H_n(e^{j\omega})\right| = 0 \quad \omega = 0 \qquad (14)$$

$$\frac{d}{d\omega}\left|H_n(e^{j\omega})\right| = n! \quad \omega = 0 \qquad (15)$$

which is consistent with the frequency response of the proposed differentiators close to the ideal DD at low frequencies as shown in Figure 3.

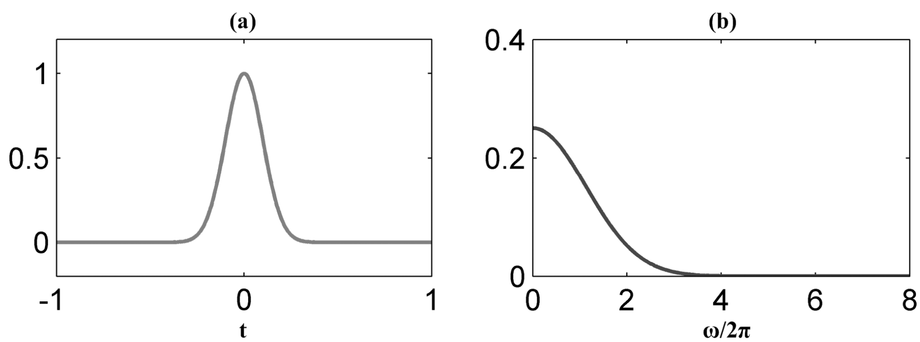

FIGURE 4: The input testing signal (a) and its Fourier transform (b).

4. SIMULATIONS AND EXPERIMENTS

4.1 The Input Testing Signal

Usually, the performance of a differentiator is evaluated by simulating Gaussian signals[5, 14, 19]. In our study, a Gaussian pulse function $g(t) = exp(-50t^2)$ sampled every 5 milliseconds, as depicted in Figure 4, was taken as the input testing signal. According to the Fourier transform of the input testing signal (Figure 4), we found that the frequency of the input signal containing the major components is maintained below 4 Hz.

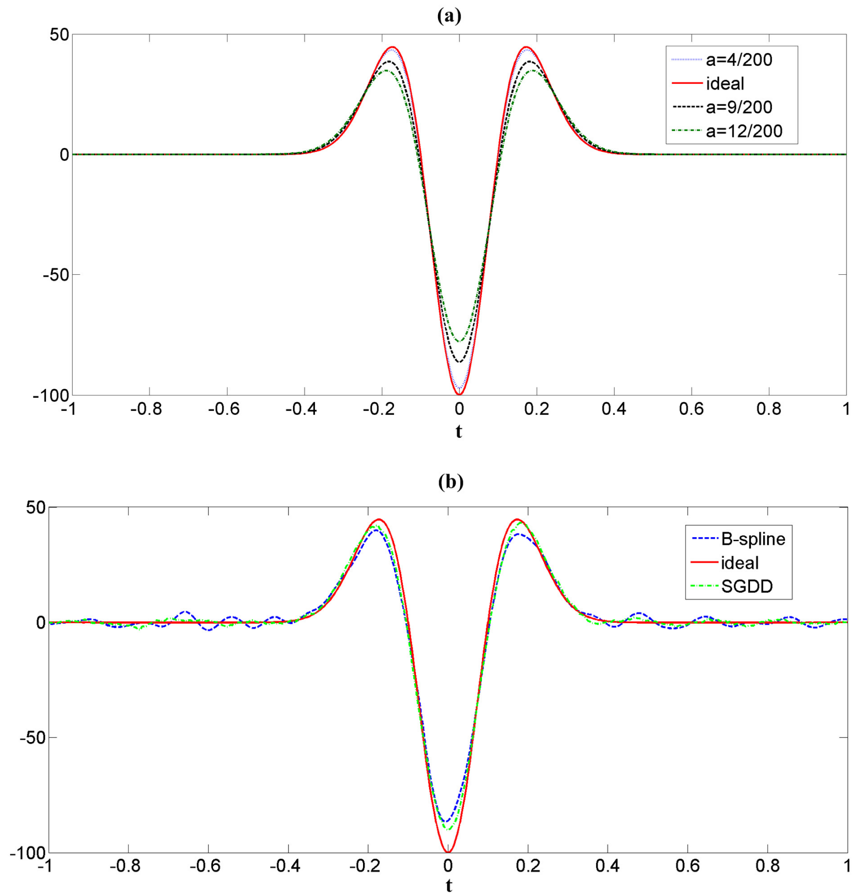

FIGURE 5: (a)The ideal 2nd derivative (—) of the input testing signal, and the 2nd derivatives using the proposed 2nd order differentiator at different values of factor *a*; (b) The ideal 2nd derivative (—), the 2nd derivative of the input testing signal contaminated by Gaussian white noise using the 2nd order Savitzky-Golay digital differentiator (SGDD) (–·), and the 2nd derivative using the 2nd order proposed differentiator (---).

Values of factor *a*	The *t* value of the first peak	the *t* value of the second peak	The first zero-crossing point	The second zero-crossing point
4/200	-35/200	35/200	-20/200	20/200
6/200	-36/200	36/200	-20/200	20/200
7/200	-36/200	36/200	-21/200	21/200
8/200	-36/200	36/200	-21/200	21/200
9/200	-36/200	36/200	-21/200	21/200
10/200	-37/200	37/200	-21/200	21/200
12/200	-38/200	38/200	-22/200	22/200

TABLE 2: The positions of characteristic points of the 2nd derivative waveform of the testing signal obtained by the proposed 2nd order differentiator.
Note: the corresponding positions of characteristic points of the ideal 2nd derivative waveform of the input testing signal respectively are -35/200, 35/200, -20/200, 20/200.

4.2 The Optimal Factor a

The method of choosing optimal factor *a* was displayed by giving an example. Using table 1, we know that the maximum value of factor *a* is 9/200 when the 3 dB cut-off frequency of the proposed 2nd order differentiator is not less than 4 Hz. Figure 5a displayed the ideal second derivative of the input testing signal and the waveforms derived by the proposed second order differentiator at different values of factor *a*. Additionally, according to Table 2, we found that it could get good signal-preservation when *a* is not more than 9/200.

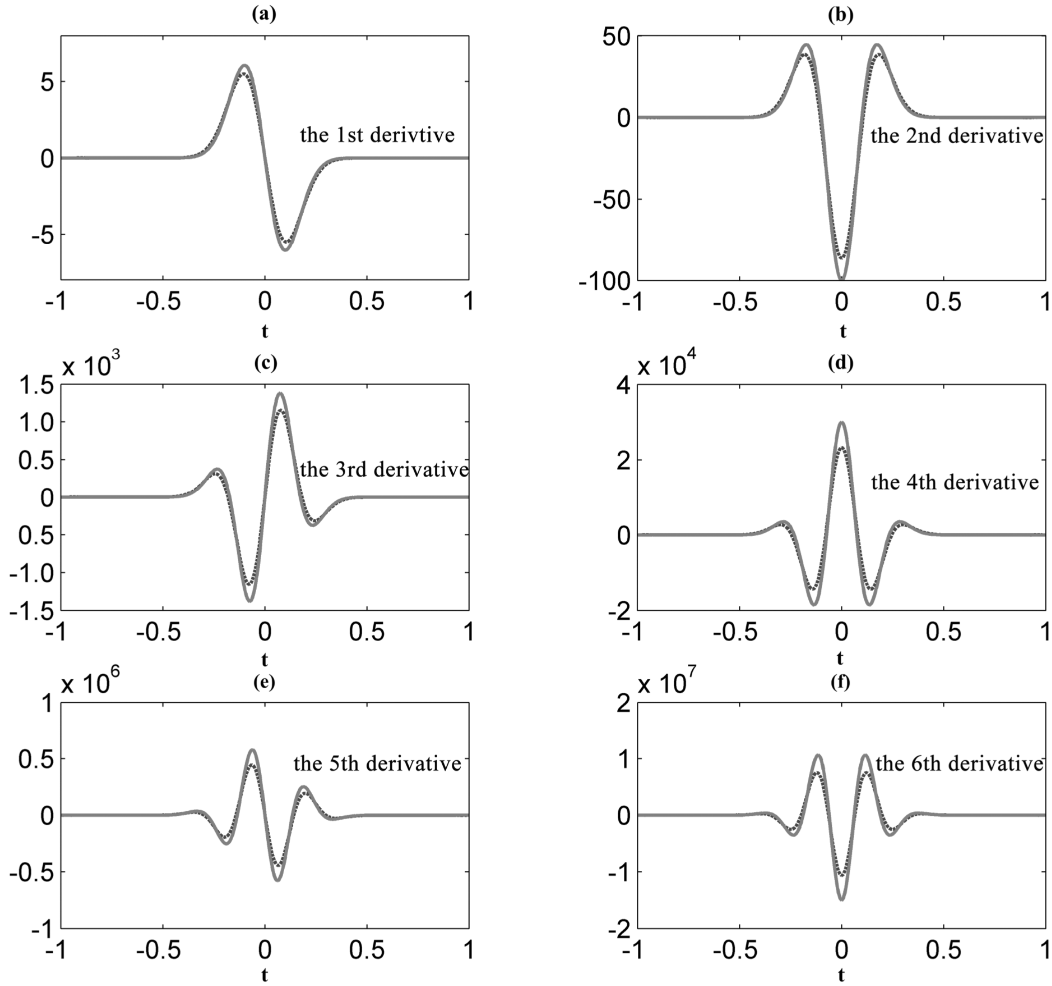

FIGURE 6: The first to sixth order derivatives of the input testing signal using the proposed differentiators (---), and corresponding ideal derivatives (—).

4.3 The Anti-Noise Capability Of The Proposed Differentiators

The anti-noise capability was evaluated by adding uncorrelated Gaussian white noise with signal-to-noise ratio (SNR) = 28.5 dB to the input testing signal. The results derived by the proposed second order differentiator at factor a = 9/200 and the second order SGDD by using fitting coefficients of fourth-order polynomials on 69 points were compared. The 3 dB cut-off frequencies of the two differentiators were both about 4.2 Hz. As can be seen from Figure 5b, both the proposed differentiator and the SGDD could restrain the high frequency noises, and the proposed differentiator got a smoother waveform than the SGDD.

4.4 The First To Sixth Derivatives Of The Input Signal

Several derivative waveforms of the input testing signal are used to validate the feasibility of the proposed differentiators. Figure 6 displayed the first to sixth derivatives of the input testing signal obtained by the proposed differentiators at the maximum value of factor *a*.

5. DISCUSSION

This study has presented a new method for designing LPDDs based on B-splines, where some examples have been used to validate the reliability and the anti-noise capability of the proposed differentiators. The proposed differentiators have some good properties, making them have some advantages in obtaining the derivatives of input signals. One is that the value of *a* could be adaptively selected by calculating the maximum allowable value of *a*. Another is that the trade-off between noise-reduction and signal preservation can be made by selecting the maximum allowable value of *a*, when the 3 dB cut-off frequency of differentiators is equal to that of the sampled signal. In addition, the cut-off frequency of the proposed differentiators is independent of sampling rate. Therefore, the proposed differentiators could be applied to a wide range of low-frequency signals.

The value of factor *a* could be adaptively selected. According to the Fourier transform of the Gaussian function, the input testing signal components cover the entire frequency range [20]. The higher the frequency is, the fewer components of the signal are distributed. Most of the signal components are maintained below 4 Hz. Therefore, when the value of *a* is low, the proposed differentiator could preserve more information of the input testing signal because of its high 3 dB cut-off frequency (as shown in Figure 2). As *a* reaches the maximum allowable value, the 3 dB cut-off frequency of differentiators is equal to or close to the maximum significant frequency of the input signal. In this case, the differentiator could filter out more of the high frequency components of the input signal, which displays by the differences of amplitudes of peaks and troughs in the two waveforms (Figure 5a). However, the differences of the positions of peaks, troughs, and the zero-crossing are little (Table 2). This illustrates the proposed differentiators could preserve signal's original features when *a* is not more than the maximum allowable value (Figure 6).

Our differentiators could easily get the trade-off between noise-reduction and signal preservation. Numerous approaches of differentiators designs have previously been displayed. the SGDD is currently one of the most common differentiators, and also is a finite impulse response LPDD [5, 12]. The SGDDs have many excellent properties [5, 21], but their frequency response have several ripples at high frequencies (Figure 1), which may affect the results of SGDDs filtering the high-frequency noises. By contrast, the frequency response of the proposed differentiators have few ripples (Figure 3), reducing almost all of the high-frequency noises. When *a* reaches the maximum allowable value, the proposed differentiator could get trade-off between noise-reduction and signal preservation. That explains the waveform derived by the proposed 2nd order differentiators is smoother than that of the 2nd order SGDD (as shown in Figure 5b). In addition, the only one key parameter of the proposed differentiator is the factor *a*, which directly determines the 3 dB cut-off frequency of differentiators. Then, using the equation between factor *a* and the 3 dB cut-off frequency, we could get the value of *a* according to the characteristics of the input signal. This makes the proposed differentiators flexible and easy to control, avoiding the work of selecting parameters of the SGDDs by testing [13].

This study proposed a easy method for designing high order LPDDs. There have been several studies on the designs of the first order LPDDs [23, 24]. However, high order LPDDs can only be constructed by cascading the first order LPDDs one by one in these studies. Therefore, the designed high order differentiators are very complicate. The proposed high order LPDD can be constructed by cascading only two low order LPDDs, and the coefficients of the differentiator can be easily acquired.

Finally, it is important to note that the cut-off frequency of the proposed differentiator is independent of sampling rate according to (5). Consequently, our differentiators are not limited by the signals with a wide range of applications. In addition, further studies in reducing the transition band of the proposed differentiator are required to improve the anti-noise capability of the differentiator.

6. CONCLUSION

The primary goal of this paper is to introduce a method for designing any order LPDDs. Several examples of the designs of the first to sixth order differentiator and some simulations were presented to validate the feasibility of this method. All these properties analysis and simulations indicate that the differentiators designed by the proposed method could be well suitable for different types of low-frequency signals, and the trade-off between noise-reduction and signal preservation could be made by selecting the maximum allowable value of *a*. But yet the behavior of the proposed differentiators needs to be tested in a wide range of situations, including the applications in reality. Further work also needs to concentrate on reducing the transition bandwidth to further improve the anti-noise capability of the proposed differentiators, especially for high order differentiators.

7. APPENDIX

7.1 Appendix A

Let $g[n]$ denote the discrete representation of $\beta_m(t)$

$$g[n] = \beta_m(n \cdot T) \qquad \text{(A.1)}$$

where T designates the sampling period. The Fourier transform of $\beta_m(t)$ and the discrete transform of $g[n]$ are

$$S(\omega) = \int_{-\infty}^{+\infty} \beta_m(t) e^{-j\omega t} dt \approx \sum_{n=-\infty}^{+\infty} \beta_m(nT) e^{-j\omega nT} / T \qquad \text{(A.2)}$$

$$G(\mathrm{e}^{j\omega}) = \sum_{-\infty}^{+\infty} g[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \beta_m(nT) e^{-j\omega n} \qquad \text{(A.3)}$$
$$= S(\omega / T) / T$$

Now let f_m, ω_c respectively denote the 3 dB cut-off frequency of $S(\omega)$ and the normalized cut-off angular frequency of $G(e^{j\omega})$.

$$|S(2\pi f_m)| = |S(0)| / \sqrt{2} \qquad \text{(A.4)}$$

$$\left|G(e^{j\omega_c})\right| = |S(\omega_c / T)| / T = S(0) / (\sqrt{2} T) \qquad \text{(A.5)}$$

Using (A.4) and (A.5), we can derive

$$\omega_c = 2\pi f_m \cdot T \qquad \text{(A.6)}$$

of which the corresponding ordinary frequency is

$$f_{3dB} = \frac{\omega_c}{2\pi} \cdot f_s = \frac{2\pi f_m \cdot T}{2\pi} \cdot \frac{1}{T} = f_m \qquad \text{(A.7)}$$

7.2 Appendix B

The Fourier transform of $\beta_m(t/a)$ is given by

$$F(\omega) = \int_{-\infty}^{+\infty} \beta_m\left(\frac{t}{a}\right) e^{-j\omega t} dt \qquad \text{(B.1)}$$
$$= a \cdot \int_{-\infty}^{+\infty} \beta_m(t) e^{-j\omega a t} dt = a \cdot S(a\omega)$$

Let f_c denote the 3 dB cut-off frequency of $F(\omega)$. Using (A.4), we derive

$$f_c \cdot a = f_m \qquad \text{(B.2)}$$

7.3 Appendix C

Parameters	Representations
$n = 1$	$\beta_3^{(1)}(t) = \begin{cases} t+1.5, -1.5 \le t \le -0.5 \\ -2t, -0.5 < t \le 0.5 \\ -t-1.5, 0.5 < t \le 1.5 \end{cases}$
$n = 2$	$\beta_4^{(2)}(t) = \begin{cases} 3\|t\| - 2, \|t\| \le 1 \\ -\|t\| + 2, 1 < \|t\| \le 2 \end{cases}$
$n = 3$	$\beta_5^{(3)}(t) = \begin{cases} t+2.5, -2.5 \le t \le -1.5 \\ -4t-5, -1.5 < t \le -0.5 \\ 6t, -0.5 < t \le 0.5 \\ -4t+5, 0.5 < t \le 1.5 \\ -t-2.5, 1.5 < t \le 2.5 \end{cases}$
$n = 4$	$\beta_6^{(2)}(t) = \begin{cases} -10\|t\| + 6, \|t\| \le 1 \\ 5\|t\| - 9, 1 < \|t\| \le 2 \\ -\|t\| + 3, 2 < \|t\| \le 3 \end{cases}$

TABLE 3: The representations of the *n*th derivative of b-spline bias functions of degree *n*+2.

7.4 Appendix D

Let $H_n(e^{j\omega})$ denote the frequency response of the proposed differentiator of degree *n* that can be written as

$$\begin{aligned} H_1(\mathrm{e}^{j\omega}) &= T^2 \cdot a^{-2} \cdot \sum_{i=-\infty}^{+\infty} \beta_m^{(1)}(iT/a)e^{-j\omega i} \\ &\approx T^2 \cdot a^{-2} \cdot S_1(a\omega/T) \cdot a/T \\ &= (\mathrm{T/a})^1 \cdot S_1(a\omega/T) \end{aligned} \qquad n = 1 \qquad \text{(D.1)}$$

$$\begin{aligned} H_{n1+n2}(\mathrm{e}^{j\omega}) &= H_{n1}\left(\mathrm{e}^{j\omega}\right) \cdot H_{n2}\left(\mathrm{e}^{j\omega}\right) \\ &= \left[T^{n1+1} \cdot a^{-n1-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m1}^{(n1)}(iT/a)e^{-j\omega i}\right] \\ &\cdot \left[T^{n2+1} \cdot a^{-n2-1} \cdot \sum_{i=-\infty}^{+\infty} \beta_{m2}^{(n2)}(iT/a)e^{-j\omega i}\right] \\ &\approx \left[T^{n1+1} \cdot a^{-n1-1} \cdot S_{n1}(a\omega/T) \cdot a/T\right] \\ &\cdot \left[T^{n2+1} \cdot a^{-n2-1} \cdot S_{n2}(a\omega/T) \cdot a/T\right] \\ &= (T/a)^n \cdot S_{n1+n2}(a\omega/T) \end{aligned} \qquad n > 1 \qquad \text{(D.2)}$$

$$S_{n1+n2}(\omega) = \int_{-\infty}^{+\infty} \beta_{m1+m2}^{(n1+n2)}(t)e^{-j\omega t}dt \qquad \text{(D.3)}$$

where *n* is the value of *n*1 and *n*2 ($n > 1$), T is the sampling period. The *n*th derivative of $\beta_m(t)$ is derived by [22]

$$\beta_m^{(n)}(t) = \sum_{i=0}^{n} (-1)^i C_n^i \beta_{m-n}(t - n/2 + i) \qquad \text{(D.4)}$$

Setting $m = m1 + m2$, $\omega = 0$, (D.3) becomes

$$
\begin{aligned}
S_{n1+n2}(\omega)\big|_{\omega=0} &= \int_{-\infty}^{+\infty} \beta_{m1+m2}^{(n1+n2)}(t)dt \\
&= \int_{m/2}^{-m/2} \sum_{i=0}^{n} (-1)^i C_n^i \beta_{m-n}(t-n/2+i)dt \\
&= \sum_{i=0}^{n} (-1)^i C_n^i \int_{-m/2}^{m/2} \beta_{m-n}(t-n/2+i)dt \\
&= \sum_{i=0}^{n} (-1)^i C_n^i \\
&= 0
\end{aligned} \tag{D.5}
$$

And the nth derivative of $S_n(\omega)$ at $\omega = 0$ is

$$
\begin{aligned}
S_{n1+n2}^{(n1+n2)}(\omega)\big|_{\omega=0} &= \int_{-\infty}^{+\infty} \beta_m^{(n)}(t)(-j\omega)^n dt \\
&= (\text{- j})^n \int_{m/2}^{-m/2} t^n d\beta_m^{(n-1)}(t) \\
&= (\text{- j})^n \left[t^n \beta_m^{(n-1)}(t)\Big|_{t=-m/2}^{t=m/2} - n \int_{m/2}^{-m/2} t^{n-1} \beta_m^{(n-1)}(t)dt \right] \\
&= (\text{- j})^n \cdot (-1)^n \cdot n! \cdot \int_{-m/2}^{m/2} \beta_m(t)dt \\
&= j^n \cdot n!
\end{aligned} \tag{D.6}
$$

Then we can get the frequency response of the proposed differentiator of degree n at $\omega = 0$, as well as the nth derivative of $H_n(e^{j\omega})$ from (D.1), (D.2), and (D.6).

$$
\begin{aligned}
H_n(\mathrm{e}^{j\omega})\big|_{\omega=0} &= (T/a)^n \cdot S_{n1+n2}(a\omega/T)\big|_{\omega=0} \\
&= 0
\end{aligned} \tag{D.7}
$$

$$
\begin{aligned}
& H_n^{(n)}(\mathrm{e}^{j\omega})\big|_{\omega=0} \\
&= (T/a)^n \cdot (a/T)^n \cdot S_{n1+n2}^{(n1+n2)}(a\omega/T)\big|_{\omega=0} = j^n \cdot n!
\end{aligned} \tag{D.8}
$$

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